Exam 2 (Total 100 points)

1. True/False (3 points each)
   a) **T** C(n, n) = 1.
   b) **F** There are 11! distinct orderings of the letters of the word MATHEMATICS.
   c) **F** Recursive algorithm is always more efficient than its iterative counterpart.
   d) **F** $1 + 10 + 100 + 1000 + ... + 10^{1000} = 10^{1001} - 1$
   e) **F** $\sum_{i=1}^{2}\sum_{j}^{5} 1 = 1$.

2. (20 points) How many strings of decimal digits (0, 1, 2,…,9) of length 10 have
   a) Exactly three 0's? **$C(10, 3)*9^7$**
   b) At least three 0's? **$10^{10} - (9^{10} + C(10, 1)*9^9 + C(10, 2)*9^8)$**
   c) Sum of all 10 digits equal to 3? **C(10, 3) + 2C(10, 2) + C(10, 1)**

3. (20 points) Give recursive definition of
   a) The set of positive integers not divisible by 5

      Initial elements: 1, 2, 3, 4 ÎS;

      Recursion: if s ÎS, then s+5 ÎS.

   b) The function that reverses a string (Hint: a string of length greater than 0 can be represented as xy where x is the first symbol of the string and y is the rest of the string. For example, for string abcd, we have x = a and y = bcd.)

      reverse(s: s is a string)

      if s is empty **then** return s;

      else { write s as xy;

      return reverse(y)x}

4. (20 points) Use mathematical induction to prove that for any integer n > 1, $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$.

   Basis step: for n = 2: $\frac{1}{1^2} + \frac{1}{2^2} = 1.25 < 2 - \frac{1}{2} = 1.5$

   Inductive step: Assume it holds for any n, show it holds for n+1.

   $\sum_{i=1}^{n} \frac{1}{i^2} < 2 - \frac{1}{n}$;

$$\sum_{i=1}^{n+1} \frac{1}{i^2} < 2 - \frac{1}{n} + \frac{1}{(n+1)^2}$$

$$= 2 - \frac{(n+1)^2 - n}{n(n+1)^2} = 2 - \frac{n^2 + n + 1}{n(n+1)^2} = 2 - \frac{n^2 + n}{n(n+1)^2} - \frac{1}{n(n+1)^2}$$

$$= 2 - \frac{n(n+1)}{n(n+1)^2} - \frac{1}{n(n+1)^2} = 2 - \frac{1}{n+1} - \frac{1}{n(n+1)^2}$$

$$< 2 - \frac{1}{n+1}$$

5. (15 points) Consider the following recurrence relation and the initial condition:

$$a_0 = 0; \quad a_n = a_{n-1} + 2 \text{ for } n \geq 1.$$

a) List the 5 elements $a_1, \cdots, a_5$ of the sequence defined by this recurrence relation.

$$a_1 = 2; \; a_2 = 4; \; a_3 = 6; \; a_4 = 8; \; a_5 = 10.$$

b) Solve this recurrence relation, i.e, find the general formula for a_n.

$$a_n = 2n.$$

6. (10 points) Let f be a function from S to T where S and T are finite sets. Show that if |S| > |T|, then f is not one-to-one.

By pigeonhole principle, at least 1 element in T must have at least 2 pre-images in S by function *f*.