Fourier transforms and series

A Fourier transform converts a function of time into a function of frequency
f is frequency in hertz
t is time in seconds $\quad t = \frac{1}{f}$ and $f = \frac{1}{t}$
$\omega = 2\pi f$
i is $\sqrt{(-1)}$
$e^{ia} = cos(a) + i \ sin(a)$

$X(f)$ is a frequency spectrum, complex value verses frequency
$x(t)$ is a signal amplitude, complex value verses time

The continuous Fourier Transform is

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-i2\pi ft} dt$$

The continuous Inverse Fourier Transform is

$$x(t) = \int_{-\infty}^{\infty} X(f) e^{i2\pi ft} df$$

$X(k)$ is a discrete frequency spectrum at frequency bin k, complex value
k is along the frequency axis
$x(n)$ is a signal amplitude at sample n, complex value
n is along the time axis
N is the number of frequency bins and the number of samples

The Discrete Fourier Transform, DFT, is

$$X(k) = \sum_{n=0}^{N-1} x(n) e^{\frac{-i2\pi kn}{N}}$$

The Inverse Discrete Fourier Transform, IDFT

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{\frac{i2\pi kn}{N}}$$

The Discrete Fourier Transform using sin and cos

$$X(k) = \sum_{n=0}^{N-1} x(n)\Big(cos(\frac{2\pi kn}{N}) - i\ \ sin(\frac{2\pi kn}{N})\Big)$$

The IDFT using sin and cos is

$$x(n) = \frac{1}{N}\sum_{n=0}^{N-1} X(n)\Big(cos(\frac{2\pi kn}{N}) + i\ \ sin(\frac{2\pi kn}{N})\Big)$$

The Fast Fourier Transform, FFT

The FFT is a numerical method for computing the DFT.
The FFT is an order $n\log_2 n$ time algorithm.
The DFT is an order n^2 time algorithm.
Both compute approximately the same values.

The Inverse Fast Fourier Transform, IFFT

The IFFT is a numerical method for computing the IDFT.
The IFFT is an order $n\log_2 n$ time algorithm.
The IDFT is an order n^2 time algorithm.
Both compute approximately the same values.

Note that the values of $X(k)$ for $k > \frac{N}{2}$, the Nyquist frequency, are alias.
In order to compute the unaliased normalized spectrum, N even, $\frac{N}{2}$ values:

for $k = 1$ to $\frac{N}{2} - 1$ $X(k) = \frac{conj((X(k)+conj(X(N-k)))}{N}$

$X(0)$ is the DC component, the average value of the signal
$X(1)$ is the complex fundamental frequency, 1 hertz for N samples in one second.
$X(\frac{N}{2} - 1)$ is the highest computed frequency $(\frac{N}{2} - 1)$ hertz
$X(1)$ is the complex fundamental frequency, 1 MHz for N samples in one microsecond,
$X(\frac{N}{2} - 1)$ is the highest computed frequency $(\frac{N}{2} - 1)$MHz

In the Fourier Series, below,
the real part of $X(k)$ will be a_k the coefficient of $cos(2\pi k)$,
the imaginary part of $X(k)$ will be b_k the coefficient of $sin(2\pi k)$.

A truncated Fourier Series may be written as

for $n = 0$ to $\frac{N}{2} - 1$

$$x(n) = \sum_{k=0}^{\frac{N}{2}-1} a_k cos(\frac{2\pi kn}{N}) + b_k sin(\frac{2\pi kn}{N})$$

The a_k are the real values of anti aliased normalized $X(k)$
The b_k are the imaginary values of the anti aliased normalized $X(k)$

Some example series:

square waves:

$$x(n) = \frac{4}{\pi} \sum_{k=odd}^{N-1} \frac{(-1)^{(k-1)/2}}{k} cos(\frac{2\pi kn}{N})$$

$a_1 = 1.273, a_3 = -0.424, a_5 = 0.255, ... \ all \ \ b_k = 0.0$

$$x(n) = \frac{4}{\pi} \sum_{n=odd}^{N-1} \frac{1}{k} sin(\frac{2\pi kn}{N})$$

$b_1 = 1.273, b_3 = 0.424, b_5 = 0.255, ... \ all \ \ a_k = 0.0$

triangle waves:

$$x(n) = \frac{8}{\pi^2} \sum_{k=odd}^{N-1} \frac{1}{k^2} cos(\frac{2\pi kn}{N})$$

$a_1 = 0.811, a_3 = 0.090, a_5 = 0.032, ... \ all \ \ b_k = 0.0$

$$x(n) = \frac{8}{\pi^2} \sum_{k=odd}^{N-1} \frac{(-1)^{(k-1)/2}}{k^2} sin(\frac{2\pi kn}{N})$$

$b_1 = 0.811, b_3 = -0.090, b_5 = 0.032, ... \ all \ \ a_k = 0.0$

saw tooth wave:

$$x(n) = \frac{2}{\pi} \sum_{k=1}^{N-1} \frac{(-1)^{k-1}}{k} sin(\frac{2\pi kn}{N})$$

$b_1 = 0.637, b_3 = -0.318, b_5 = 0.212, ... \ all \ \ a_k = 0.0$

Modulation of a carrier by a signal:

The mathematical definition of modulation is derived from basic relations:

$$sin(A+B) = sin(A)cos(B) + cos(A)sin(B)$$

$$cos(A+B) = cos(A)cos(B) - sin(A)sin(B)$$

$$sin(A-B) = sin(A)cos(B) - cos(A)sin(B)$$

$$cos(A-B) = cos(A)cos(B) + sin(A)sin(B)$$

$m(t)$ is the continuous modulation signal. Typically in range -1.0 to +1.0
$m(n)$ is the sampled modulation signal. Typically N samples.
f_c is the continuous carrier frequency.
$sin(2\pi f_c t)$ is the continuous carrier signal.
$sin(\frac{2\pi n}{N_c})$ is the sampled carrier signal.
(not covered here is down conversion to an intermediate frequency
and narrow band filtering, that provides noise reduction.)

Amplitude Modulation, AM, for a carrier A and modulation B is:
$modulated\ signal = sin(A)(1.0 + sin(B))$ and either sin may be cos .
The resulting spectrum of the $modulated\ signal$ has $sin(A)$ and
$cos(A+B)$ and $cos(A-B)$, known as a double sideband signal.
The same equations apply, taking the signal B as a $B(k)$ transform of $b(n)$.
For numeric computation the time domain $modulated\ signal$ is represented as
$x(n) = sin(\frac{2\pi n}{N_c})(1.0 + m(n))$ where $m(n)$ is the time domain sampled modulation.
The demodulation is performed using:
$sin(B)$ is computed approximately as lowpassfilter$(abs(modulated\ signal) - 1.0)$

Frequency Modulation, FM, for a carrier A and modulation B is:
$modulated\ signal = sin(A + scale \int sin(B))$ and either sin may be cos .
The continuous FM signal is $sin(2\pi f_c\ t + scale \int m(t)dt)$.
The $scale$ determines the band width of the modulated signal.
The demodulation may be performed using several techniques including
discriminator and ratio detector to determine the instantaneous frequency.

Phase Modulation, PM, for a carrier A and analog modulation B is:
$modulated\ signal = sin(A + scale\ \ sin(B))$ and either sin may be cos .
The continuous PM signal is $sin(2\pi f_c\ t + scale\ \ m(t))$.
The $scale$ determines the band width of the $modulated\ signal$.
The demodulation may be performed using $arcsin$ of the measured phase difference from the carrier reference.

Frequency Shift Modulation for a carrier A and modulation B is:
$modulated\ signal = sin(A + scale\ \ sin(B))$ and either sin may be cos .
$m(t)$ may be analog, although typically used for digital modulation as FSK.
The continuous FSM signal is $sin(2\pi(f_c + scale\ \ m(t))t)$.
For numeric computation the time domain $modulated\ signal$ is represented as $x(n) = sin(\frac{2\pi(n+scale\ \ m(n))}{N_c})$ where $m(n)$ is the time domain sampled modulation.
The $scale$ determines the band width of the $modulated\ signal$.
The demodulation may be performed using several techniques. One technique is to measure the time between zero crossings of the modulated signal. Subtract $\frac{0.5}{f_c}$ from each time and low pass filter, integrate, the resulting signal, then unscale. This approximately reconstructs the original modulation $m(n)$.

Quadrature Phase Shift Keying, QPSK, is used to send a symbol, two bits in this case, for a few cycles,j, then another symbol for a few cycles,j, etc.
Typical transmission uses modulation f_m in the KHz range with
$\Phi = \frac{\Pi}{4}$ for 00_2,
$\Phi = \frac{3\pi}{4}$ for 01_2,
$\Phi = -\frac{\pi}{4}$ for 10_2,
$\Phi = -\frac{3\pi}{4}$ for 11_2
The continuous QPSK signal is $sin(2\pi f_c\ t) * sin(2\pi f_m\ t + \Phi)$.
Demodulation converts the continuous signal to in-phase, I and $\frac{\pi}{2}$ quadrature, Q, signals at frequency f_m .
For the known few cycles,j, $Isum = \Sigma sin(2\pi f_m\ t) \times I$ and $Qsum = \Sigma sin(2\pi f_m\ t) \times Q$, then
$Isum > 0$ and $Qsum > 0$ yeilds symbol 00_2
$Isum < 0$ and $Qsum > 0$ yeilds symbol 01_2
$Isum > 0$ and $Qsum < 0$ yeilds symbol 10_2
$Isum < 0$ and $Qsum < 0$ yeilds symbol 11_2

Single Side Band modulation, SSB, results from the equations:
Computing the product of signal A with signal B
$(cos(A)+i\ \ sin(A))(cos(B)+i\ \ sin(B)) = cos(A)cos(B)-sin(A)sin(B)+i(sin(A)cos(B)+cos(A)sin(B)) = cos(A+B)+i\ \ sin(A+B)$
results in the sum frequency signal $A+B$.
This is called single side band modulation, producing the upper sideband.

Computing the product of conjugate signal A with signal B
$(cos(A)-i\ \ sin(A))(cos(B)+i\ \ sin(B)) = cos(A)cos(B)+sin(A)sin(B)-i(sin(A)cos(B)-cos(A)sin(B)) = cos(A-B)-i\ \ sin(A-B)$
results in the difference frequency signal $A-B$.
This is called single side band modulation, producing the lower sideband.

Computing the product of conjugate signal A with signal $A+B$
$(cos(A)-i\ \ sin(A))(cos(A+B)+i\ \ sin(A+B)) = cos(A)cos(A+B)+sin(A)sin(A+B)-i(sin(A)cos(A+B)-cos(A)sin(A+B)) = cos(A-(A+B))-i\ \ sin(A-(A+B)) = cos(B)+i\ \ sin(B)$
results in the demodulation, reproducing the signal B.

Computing the product of signal A with signal $A-B$
$(cos(A)+i\ \ sin(A))(cos(A-B)-i\ \ sin(A-B)) = cos(A)cos(A-B)+sin(A)sin(A-B)+i(sin(A)cos(A-B)-cos(A)sin(A-B)) = cos(A-(A-B))+i\ \ sin(A-(A-B)) = cos(B)+i\ \ sin(B)$
results in the demodulation, reproducing the signal B.

In general the signal B will contain many frequencies at various phase angles.
The same equations apply, taking the signal B as a $B(k)$ transform of $b(n)$.
For numeric computation the time domain carrier is represented as

$$cos(\frac{2\pi n}{N_c})+isin(\frac{2\pi n}{N_c})$$

and the modulation signal $m(n)+i\ \ m'(n)$ where $m'(n)$ is $m(n)$ phase shifted -90 degrees, and N_c determines the carrier frequency.