Finite Element Method over triangles

One method for numerically solving partial differential equation boundary value problems is the Finite Element Method, FEM, and specifically the Galerkin method.

Given a two dimensional linear partial differential equation with dependent variable u and independent variables x, y

$$L\left(u(x,y)\right) = f(x,y)$$

L is a general linear differential operator of some specific order. Samples for up to fourth order are shown below.

Find the approximate solution at vertices $U(x,y)$ numerically, that has boundary conditions chosen that make the problem of finding $u(x,y)$ well posed on the domain Ω for all $(x,y) \in \Omega$.

We denote the approximated solution $U(x_i, y_i)$ as U_i at vertex (x_i, y_i).

We take $i = 1 \dots n$ for specific vertices $x_1, y_1, \dots, x_n, y_n$. These are vertices of a properly trianglated region Ω covered by $\Omega_1 \dots \Omega_m$

Let $U(x,y) = \sum_{i=1}^{n} U_i \phi_i(x,y)$

We will use $\phi_i(x,y)$ as function about x_i, y_i as defined below, where $\phi_i(x,y) = \sum \phi_i(T,x,y)$ for all triangles, T, with area Ω_v containing vertex x_i, y_i

The Galerkin Method states:

$$\int_\Omega L\left(U(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

Substituting for $U(x,y)$ yields

$$\int_\Omega L\left(\sum_{i=1}^{n} U_k \phi_i(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

Bringing the summation out of the integral yields

$$\sum_{i=1}^{n} U_i \int_\Omega L\left(\phi_i(x,y)\right) \phi_l(x,y) dx\, dy = \int_\Omega f(x,y) \phi_l(x,y) dx\, dy$$

$L(\phi_i(x,y))$ means a substitution in $L(u(x,y))$ where $u(x,y)$ becomes $\phi_i(x,y)$, $ux(x,y)$ becomes $\phi'_{xi}(x,y)$, $uy(x,y)$ becomes $\phi'_{yi}(x,y)$, $uxy(x,y)$ becomes $\phi'_{xyi}(x,y)$, $uxx(x,y)$ becomes $\phi''_{xi}(x,y)$, $uyy(x,y)$ becomes $\phi''_{yi}(x,y)$, etc.

Writing the above in matrix form using the index $k = (i-1) \times ny + j$ for rows and index $l = (i-1) \times ny + j$ for columns yields

$$\begin{vmatrix} \int_\Omega L\left(\phi_1(x,y)\right)\phi_1(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_1(x,y)dx\,dy & \dots & \int_\Omega L\left(\phi_n(x,y)\right)\phi_1(x,y)dx\,dy \\ \int_\Omega L\left(\phi_1(x,y)\right)\phi_2(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_2(x,y)dx\,dy & \dots & \int_\Omega L\left(\phi_n(x,y)\right)\phi_2(x,y)dx\,dy \\ & & \dots & \\ \int_\Omega L\left(\phi_1(x,y)\right)\phi_n(x,y)dx\,dy & \int_\Omega L\left(\phi_2(x,y)\right)\phi_n(x,y)dx\,dy & \dots & \int_\Omega L\left(\phi_n(x,y)\right)\phi_n(x,y)dx\,dy \end{vmatrix} \times$$

$$\begin{vmatrix} U_1 \\ U_2 \\ \dots \\ U_n \end{vmatrix} = \begin{vmatrix} \int_\Omega f(x,y)\phi_1(x,y)dx\,dy \\ \int_\Omega f(x,y)\phi_2(x,y)dx\,dy \\ \dots \\ \int_\Omega f(x,y)\phi_n(x,y)dx\,dy \end{vmatrix}$$

Note that the above applies for the “internal” non boundary nodes.

Given Dirichlet boundary values, e.g. v_1 at (x_1, y_1) and v_n at (x_{nx}, y_{ny}) the first and last rows of the above matrix equation would be:

$$\begin{vmatrix} 1 & 0 & \dots & 0 \\ & & \dots & \\ 0 & 0 & \dots & 1 \end{vmatrix} \times \begin{vmatrix} U_1 \\ \dots \\ U_n \end{vmatrix} = \begin{vmatrix} v_1 \\ \dots \\ v_n \end{vmatrix}$$

The many boundary rows may be eliminated and a $(nx-2)(ny-2)$ system of equations are solved to find the U_k for $i = 2 \dots nx-1, j = 2 \dots ny-1$

Note that, in general, numerical integration is required to compute the matrix elements and the right hand side vector elements.

Lagrange polynomials over triangles

For triangles, using $\phi_i(T[x_i, y_i, x_j, y_j, x_k, y_k])$ to determine Lagrange polynomials:

$$\phi_i(T, x, y) = c_0 + c_1 x + c_2 y$$

where

where $\phi_i(T, x, y)$ is the polynomial in a set of polynomials such that:

$$\phi_i(T, x, y) = \begin{cases} 1 \; for \; x = x_i \; y = y_i \\ 0 \; for \; x = x_j \; y = y_j \\ 0 \; for \; x = x_k \; y = y_k \end{cases}$$

solve for c_0, c_1, c_2

$$\begin{vmatrix} 1 \; x_i \; y_i \\ 1 \; x_j \; y_j \\ 1 \; x_k \; y_k \end{vmatrix} \times \begin{vmatrix} c_0 \\ c_1 \\ c_2 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \end{vmatrix}$$

Derivative with respect to x $\phi'_{xi}(x, y) = c_1$, derivative with respect to y $\phi'_{yi}(x, y) = c_2$.

These ϕ functions are only useful when only the first derivative of ϕ is needed. Second derivatives and higher are all zero.

Using the midpoint of each side of the triangle $x_{ij} = \frac{(x_i + x_j)}{2}$ and $y_{ij} = \frac{(y_i + y_j)}{2}$

$$\phi_i(T, x, y) = c_0 + c_1 x + c_2 y + c_3 x^2 + c_4 xy + c_5 y^2$$

where

where $\phi_i(T, x, y)$ is the polynomial in a set of polynomials such that:

$$\phi_i(T, x, y) = \begin{cases} 1 \; for \; x = x_i \; y = y_i \\ 0 \; for \; x = x_j \; y = y_j \\ 0 \; for \; x = x_k \; y = y_k \\ 0 \; for \; x = x_{ij} \; y = y_{ij} \\ 0 \; for \; x = x_{jk} \; y = y_{jk} \\ 0 \; for \; x = x_{ki} \; y = y_{ki} \end{cases}$$

solve for $c_0, c_1, c_2, c_3, c_4, c_5$

$$\begin{vmatrix} 1 \; x_i \; y_i \; {x_i}^2 \; x_i y_i \; {y_i}^2 \\ 1 \; x_j \; y_j \; {x_j}^2 \; x_j y_j \; {y_j}^2 \\ 1 \; x_k \; y_k \; {x_k}^2 \; x_k y_k \; {y_k}^2 \\ 1 \; x_{ij} \; y_{ij} \; {x_{ij}}^2 \; x_{ij} y_{ij} \; {y_{ij}}^2 \\ 1 \; x_{jk} \; y_{jk} \; {x_{jk}}^2 \; x_{jk} y_{jk} \; {y_{jk}}^2 \\ 1 \; x_{ki} \; y_{ki} \; {x_{ki}}^2 \; x_{ki} y_{ki} \; {y_{ki}}^2 \end{vmatrix} \times \begin{vmatrix} c_0 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \\ c_5 \end{vmatrix} = \begin{vmatrix} 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{vmatrix}$$

Higher order ϕ functions may be defined by

$$\phi_i(T, x, y) = \frac{((x - x_j)^2 + (y - y_j)^2)((x - x_k)^2 + (y - y_k)^2)}{((x_i - x_j)^2 + (y_i - y_j)^2)((x_i - x_k)^2 + (y_i - y_k)^2)}$$

$$\phi_i(T, x, y) = \frac{((x - x_j)^4 + (y - y_j)^4)((x - x_k)^4 + (y - y_k)^4)}{((x_i - x_j)^4 + (y_i - y_j)^4)((x_i - x_k)^4 + (y_i - y_k)^4)}$$

Fig 1. the three linear phi functions for one triangle

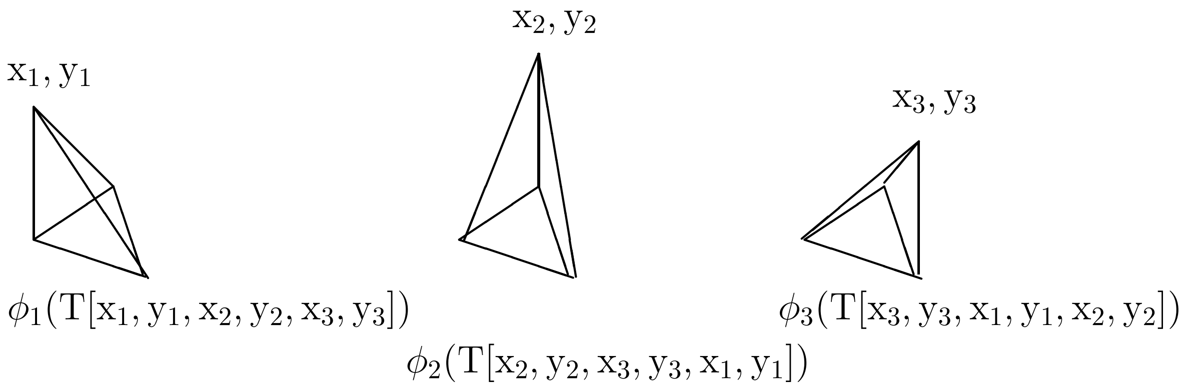
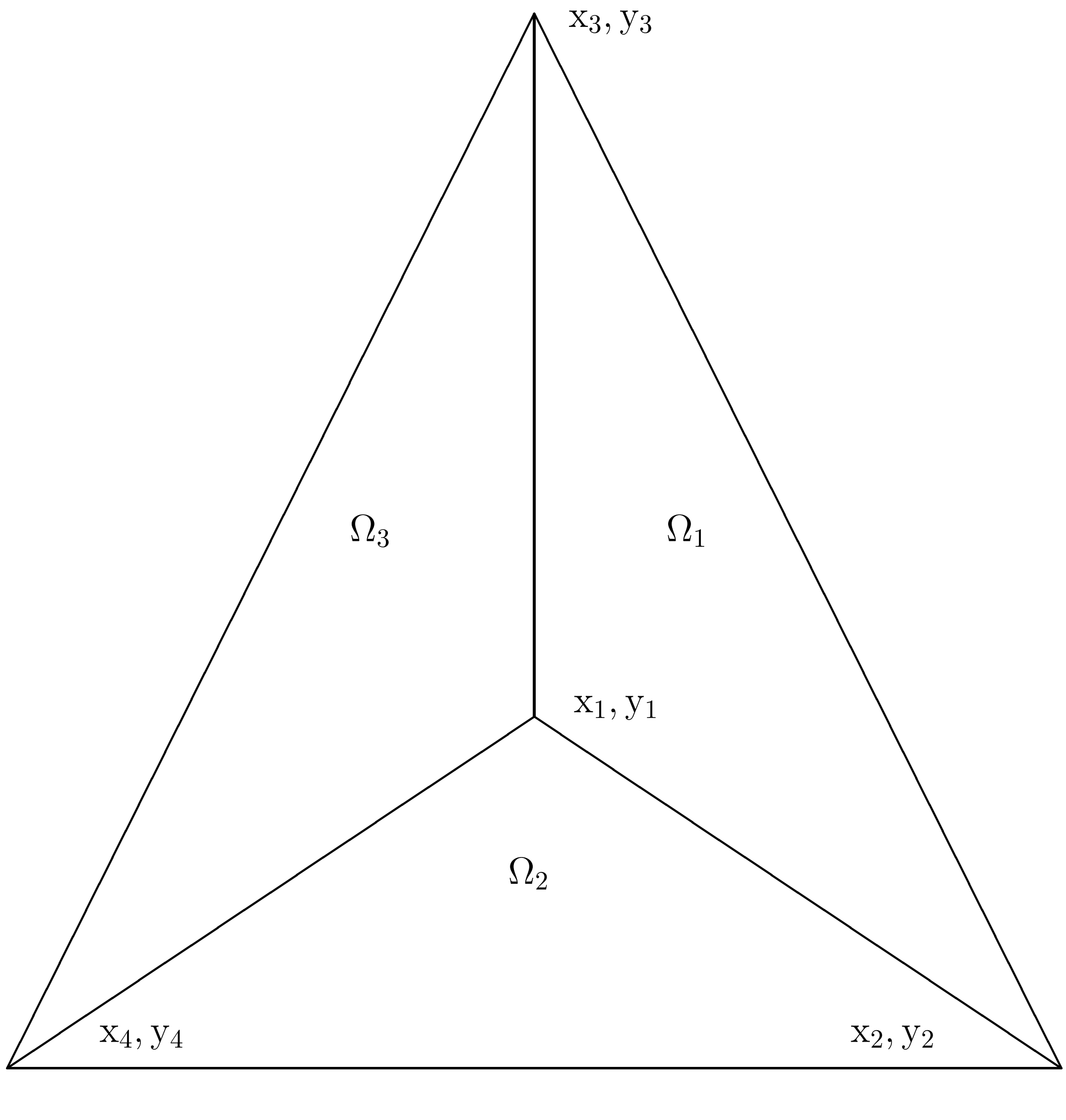


Fig 2. Top view of a triangularazation



Galerkin test functions for second order PDE
Second order ϕ functions may be defined by
see file tri_basis.h and tri_basis.c