A 3-D Spectral Integral Method (SIM) for Surface Integral Equations

Qing Huo Liu, *Fellow, IEEE*, Yun Lin, Jianguo Liu, Joon-Ho Lee, and Ergün Şimşek

***Abstract*—An efficient 3-D spectral integral method (SIM) has been proposed to speed up the method of moments (MOM) calculation of induced currents on a cuboid. This method utilizes the Toeplitz structure in the impedance matrix and the fast Fourier transform to accelerate the MOM solution. It reduces the memory and CPU time per iteration from $O(N^2)$ in the MOM to $O(N^{1.5})$ and $O(N^{1.5}\log N)$, respectively. Thus, the SIM can be also used as an efficient radiation boundary condition for the finite element method. Numerical results confirm the effectiveness of this method.**

***Index Terms*—Fast Fourier transform (FFT), finite element–boundary integral (FEM-BI) method, method of moments (MOM), spectral integral method (SIM).**

I. Introduction

Surface integral equations in electromagnetic scattering problems have traditionally been solved by the method of moments (MOM) [1]. However, it is well known that the MOM solution for surface integral equations is inefficient as it requires $O(N^2)$ memory, and $O(N^2)$ CPU time per iteration in an iterative solver for the matrix equation, where N is the number of surface unknowns. The development of the fast multipole method [2], [3] and adaptive integral method [4] significantly improves the computational efficiency of MOM solution to surface integral equations.

We propose an alternative approach, the spectral integral method (SIM), for solving 3-D surface integral equations on a cuboid [9]. This method makes use of the Toeplitz structure of the impedance matrix on the cuboid surface to apply the fast Fourier transform (FFT) algorithm for accelerating matrix-vector multiplications. It thus reduces the memory cost to $O(N^{1.5})$ and CPU time to $O(N^{1.5}\log N)$ per iteration, significantly more efficient than the MOM method.

As an application, the surface integral equations can be used as an exact radiation boundary condition for the finite element method, as in the hybrid finite-element/boundary integral (FEM-BI) method [8]. This hybrid method is effective in that arbitrary inhomogeneities can be modeled and the radiation boundary condition is highly accurate. In the special case of a flat surface (such as a cavity-backed aperture, the fast Fourier transform (FFT) algorithm can be used to speed up this computation as in the conjugate-gradient (CG) FFT algorithm [5]–[7]. However, for general 3-D surfaces, this FFT acceleration has not been developed, although in principle this can be achieved by using the fast multipole method [2], [3] and adaptive integral method [4]. We have also hybridized the SIM with the FEM.

Manuscript received September 08, 2008; revised October 20, 2008. Current version published February 11, 2009. This work was supported in part by the U.S. National Science Foundation under Grant CCF-0621862.

Q. H. Liu, Y. Lin, J. Liu, and J.-H. Lee are with the Department of Electrical and Computer Engineering, Duke University, Durham, NC 27708 USA (e-mail: qhliu@ee.duke.edu).

E. Şimşek was with the Department of Electrical and Computer Engineering, Duke University, Durham, NC 27708 USA and is now with the Department of Electrical and Electronics Engineering, Bahcesehir University, Istanbul, Turkey.

Color versions of one or more of the figures in this paper are available online at http://ieeexplore.ieee.org.

Digital Object Identifier 10.1109/LMWC.2008.2011305

II. Formulation

It is well known that the electromagnetic scattering of a homogeneous object in a homogeneous background medium can be formulated through the electric field integral equations or magnetic field integral equations [1]. In particular, the electric field integral equation for the exterior problem is

$$\hat{n}\times\mathbf{E}^{\rm inc}(\mathbf{r})=-\mathbf{M}(\mathbf{r})+\hat{n}\times\int_S\Big[jk_b\eta_b g_b(\mathbf{r}-\mathbf{r}')\mathbf{J}(\mathbf{r}')+\frac{j\eta_b}{k_b}\nabla g_b(\mathbf{r}-\mathbf{r}')\nabla'\cdot\mathbf{J}(\mathbf{r}')+\nabla g_b(\mathbf{r}-\mathbf{r}')\times\mathbf{M}(\mathbf{r}')\Big]ds(\mathbf{r}') \quad (1)$$

where $j=\sqrt{-1}; k_b, \eta_b$, and g_b are the wavenumber, intrinsic impedance, and Green's function of the background medium, respectively; $\mathbf{J}$ and $\mathbf{M}$ are the equivalent surface electric and magnetic current densities, respectively. A similar electric field integral equation can be written for the interior problem but is omitted here for brevity.

Using the conventional MOM, the integral equations can be discretized with basis functions for the unknown electric and magnetic current densities

$$\mathbf{J}(\mathbf{r})=\sum_{n=1}^{N_j}j_n\mathbf{f}_n(\mathbf{r}),\quad \mathbf{M}(\mathbf{r})=\sum_{n=1}^{N_m}m_n\mathbf{f}_n(\mathbf{r}) \quad (2)$$

where j_n and m_n are the unknown expansion coefficients of electric and magnetic current densities and $\{\mathbf{f}_n\}$ are vector basis functions such as the rooftop (RWG) basis functions. In general, we choose the same number of unknowns for the electric and magnetic current densities such that $N_j = N_m$, or the total number of unknowns is $N=2N_j$.

To reduce the memory and CPU time requirements of $O(N^2)$ in an iterative MOM solution of the resultant system matrix, here we propose an alternative method for surface integral equations on a cuboid of dimensions $L_x\times L_y\times L_z$ by using the FFT algorithm. We divide the cuboid surface with $N_x\times N_y\times N_z$ uniform elements, each element having a size $h_i=L_i/N_i$ in the i-th direction $(i=x,y,z)$. The RWG basis functions associated with the current densities in the ith direction $(i=x,y,z)$

$$\mathbf{f}_n^{(i)}(\mathbf{r})=\begin{cases}\hat{i}(1-|i|/h_i), & -h_i\le i<h_i\\ 0, & \text{otherwise.}\end{cases} \quad (3)$$

Using these basis functions in the standard MOM for (1), we obtain an algebraic equation for the exterior problem

$$[\mathbf{A}+\mathbf{B}]\mathbf{J}+\mathbf{C}\mathbf{M}=\mathbf{V} \tag{4}$$

where, with slight abuse of notations, $\mathbf{J}$ and $\mathbf{M}$ denote the column vectors representing the unknown coefficients $\{j_n\}$ and $\{m_n\}$, respectively; $\mathbf{V}$ is the source vector; and the impedance sub-matrices $\mathbf{A}, \mathbf{B}, \mathbf{C}$ (sub-matrices of $\mathbf{Z}$) are

$$A_{mn} = jk_b\eta_b \int_S ds(\mathbf{r})\mathbf{f}_m^{(i_m)}(\mathbf{r}) \cdot \int_S ds(\mathbf{r}')g_b(\mathbf{r}-\mathbf{r}')\mathbf{f}_n^{(i_n)}(\mathbf{r}')$$

$$B_{mn} = \frac{\eta_b}{jk_b} \int_S ds(\mathbf{r})\nabla \cdot \mathbf{f}_m^{(i_m)}(\mathbf{r}) \times \int_S ds(\mathbf{r}')g_b(\mathbf{r}-\mathbf{r}')\nabla' \cdot \mathbf{f}_n^{(i_n)}(\mathbf{r}')$$

$$C_{mn} = \frac{\eta_b}{jk_b} \int_S ds(\mathbf{r})\mathbf{f}_m^{(i_m)}(\mathbf{r}) \cdot \left[-\frac{1}{2}\mathbf{f}_n^{(i_n)}(\mathbf{r}) + \int_S ds(\mathbf{r}')\nabla g_b(\mathbf{r}-\mathbf{r}') \times \mathbf{f}_n^{(i_n)}(\mathbf{r}') \right].$$

If the same RWG basis functions are used for both $\mathbf{J}$ and $\mathbf{M}$ on the cuboid surfaces, the total number of unknowns is

$$N = 8(N_xN_y + N_xN_z + N_yN_z). \tag{5}$$

The total impedance matrix $\mathbf{Z}$ has the dimensions of $N \times N$. The evaluation procedure for an impedance element is identical to the conventional MOM; in particular, the self-interaction term has been evaluated using the Duffy transformation. However, the SIM only needs to evaluate a small number of MOM impedance matrix elements, as explained below.

Given the uniform surface elements on the cuboid, there are some special Toeplitz structures in these impedance matrices because of the shift invariance of the Green's function for a homogeneous background medium. We will just consider the structure of matrix $\mathbf{A}$ because matrices $\mathbf{B}$ and $\mathbf{C}$ are similar. Obviously, under the uniform mesh, the impedance matrix element A_{mn} is a function of the distance $R_{m,n}$ between the observation and source points. In general

$$R_{m,n} = R_{m_x,m_y,m_z;n_x,n_y,n_z} \tag{6}$$

where $m = (m_x, m_y, m_z)$ and $n = (n_x, n_y, n_z)$ are the compound indices for the observation and source points, respectively. Noting that the unknown current on an edge is shared by two orthogonal surfaces, we have $m_i, n_i = 1, 2, \ldots, N_i$. Depending on which surfaces of the cuboid the observation point $\mathbf{r}$ and source point $\mathbf{r}'$ are located, under the uniform mesh, the impedance matrix has three different structures as detailed:

Case 1: Source and Observation Points on Orthogonal Surfaces: In this case, the distance between the source and observation points depends on the difference in one of the indices, either $m_x - n_x$, or $m_y - n_y$, or $m_z - n_z$, depending on whether the shared direction is x, y, or z directions. For example, if the source elements are on the top xy plane at $z = L_z/2$, and the observation elements are on the xz plane at $y = L_y/2$, matrix element A_{mn} and the distance $R_{m,n}$ are

$$\begin{aligned} A_{mn} &= A^{(xz,xy)}(R_{m_x-n_x,n_y,m_z}) \\ R^2_{m_x-n_x,n_y,m_y} &= (m_x-n_x)^2h_x^2 + (N_y-n_y)^2h_y^2 \\ &\quad + (m_z - N_z)^2h_z^2. \end{aligned} \tag{7}$$

In this case, only one direction (x direction) is Toeplitz for the impedance matrix. Thus, the FFT algorithm can be applied to one direction to speed up the matrix-vector multiplications. In this example, the memory requirement is $O(N_x[N_y^2 + N_z^2])$, while the CPU time requirement is $O(N_x[N_y^2 + N_z^2]\log N_x)$.

Case 2: Source and Observation Points on the Same or Parallel Surfaces: For the source and observation points on the bottom and top surfaces (xy surfaces) away from the edges

$$\begin{aligned} A_{mn} &= A^{(xy,xy)}\left(R_{m_x-n_x,m_y-n_y}\right), \\ R^2_{m_x-n_x,m_y-n_y} &= (m_x-n_x)^2h_x^2 + (m_y-n_y)^2h_y^2 \\ &\quad + L_z^2\left(1-\delta_{m_z,n_z}\right). \end{aligned} \tag{8}$$

Note that $L_z^2(1-\delta_{m_z,n_z})$ is zero if the source and observation elements are on the same surface (i.e., $m_z = n_z$), and is L_z^2 if they are on opposite surfaces (i.e., $m_z \neq n_z$). Obviously, for this case the impedance matrix is Toeplitz in two directions (x and y directions). Thus, the FFT algorithm can be applied to two directions to speed up matrix-vector multiplications. In this example, the memory requirement is $O(N_xN_y)$, while the CPU time requirement is $O(N_xN_y\log[N_xN_y])$.

Case 3: Edge Effects—Source and/or Observation Points on an Edge of a Surface: The basis and testing functions on an edge are different from those away from edges since in this case the currents are no longer planar; they now represent current going from one plane to the adjacent orthogonal plane. Therefore, the impedance matrix has to be modified from the above impedance matrix in Case 1 and Case 2 with the Toeplitz properties. Specifically, for the example in Case 1 above, the edge effects will require an additional correction matrix which is sparse and requires $O(N_xN_z^2 + N_x^2N_z)$ memory and $O(N_xN_z^2 + N_x^2N_z)$ CPU time. Similarly, for the example in Case 2 above, the edge effects will require an additional correction matrix which is sparse and requires $O(N_xN_y^2 + N_x^2N_y)$ memory and $O(N_xN_z^2\log N_x + N_x^2N_z\log N_z)$ CPU time.

From the above discussions, the total impedance matrix can be written in three parts

$$\mathbf{Z} = \mathbf{Z}_T^{(1)} + \mathbf{Z}_T^{(2)} + \mathbf{Z}_R \tag{9}$$

where $\mathbf{Z}_T^{(1,2)}$ are the Toeplitz parts of the system matrix (the superscript indicates 1-D and 2-D Toeplitz matrices), and $\mathbf{Z}_\mathbf{R}$ is the remainder part of the system matrix which is highly sparse. Then the matrix-vector multiplication for the Toeplitz matrices can be obtained through FFT as

$$\begin{aligned} \left[\mathbf{Z}_T^{(1)} + \mathbf{Z}_T^{(2)}\right]\mathbf{x} &= \mathbf{FFT}_1^{-1}\left[\mathbf{FFT}_1\left(\mathbf{Z}_T^{(1)}\right)\mathbf{FFT}_1(\mathbf{x})\right] \\ &\quad + \mathbf{FFT}_2^{-1}\left[\mathbf{FFT}_2\left(\mathbf{Z}_T^{(2)}\right)\mathbf{FFT}_2(\mathbf{x})\right] \end{aligned} \tag{10}$$

where $\mathbf{FFT}_1$ and $\mathbf{FFT}_2$ denote the one- and two-dimensional FFT operations, respectively. Note that zero-padding is required as in the CG-FFT algorithm [5].

For the special case where $N_x = N_y = N_z$, the above Toeplitz matrix-vector multiplication requires $O(N^{1.5})$ memory and $O(N^{1.5}\log N)$ CPU time. For the remainder matrix, the corresponding cost is $O(N^{1.5})$ memory and $O(N^{1.5})$ CPU time. This computational complexity is significantly better than the MOM with $O(N^2)$ for memory and CPU time. It is similar to the fast multipole method with the one-level implementation and the adaptive integral method

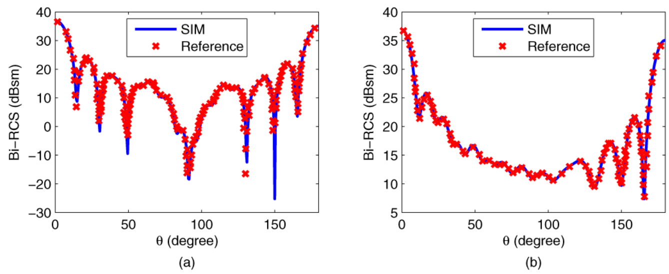

Fig. 1. Comparison of the bistatic RCS from SIM and MOM [10] at (a) $\phi = 0^\circ$ and (b) $\phi = 90^\circ$ for a 4 m PEC cube under a 0.3 GHz plane wave incident along the z direction with x-polarization.

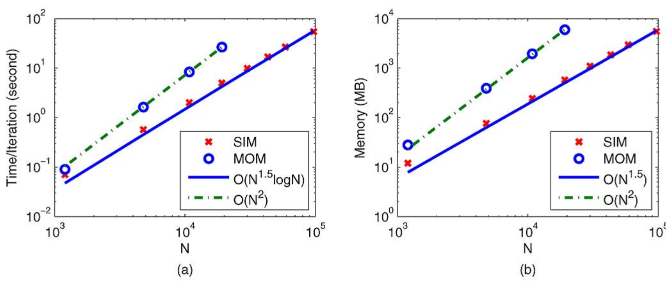

Fig. 2. CPU time (a) and memory (b) usage in the SIM and MOM.

for surface integral equations. The advantage of the spectral integral method is that there is no approximation involved; its disadvantage is that it is only applicable to surface integral equations on a cuboid.

III. Numerical Results

To validate the 3-D SIM we first show an example of plane wave scattering from a perfect electric conductor (PEC) cube of side length 4 m, a case also studied by [10]. The plane wave is incident along the z direction with x-polarization, and the operating frequency is 0.3 GHz. Figs. 1(a) and (b) compare bistatic radar cross section (RCS) results obtained by the SIM and the reference results in [10] for $\phi = 0^\circ$ and $\phi = 90^\circ$, respectively. Excellent agreement has been obtained.

The computational complexity of the SIM is studied by varying N, the number of unknowns. The CPU time and memory costs in the SIM and MOM are shown in Fig. 2(a) and (b), indicating that the SIM has a CPU time complexity of $O(N^{1.5}\log N)$, and memory complexity of $O(N^{1.5})$, compared to the $O(N^2)$ complexity for MOM in both CPU time and memory. In particular, we find that the CPU time for MOM is about six times higher for $N = 20\,000$. The SIM acceleration factor increases rapidly for larger problems.

One important application of the spectral integral method is its usage as an exact radiation boundary condition for the finite-element method and other partial-differential equation methods. Here, the SIM has been hybridized with the finite element method so that an inhomogeneous medium can be accurately modeled in the interior domain. Fig. 3 shows the RCS of a PEC cube of side length 1 m coated by a dielectric material ($\epsilon_r = 2.5$) of thickness 0.25 m on each face. The bistatic RCS results from the hybrid SIM-FEM technique for $\phi = 0^\circ$ and $\phi = 90^\circ$ for this coated PEC cube are compared

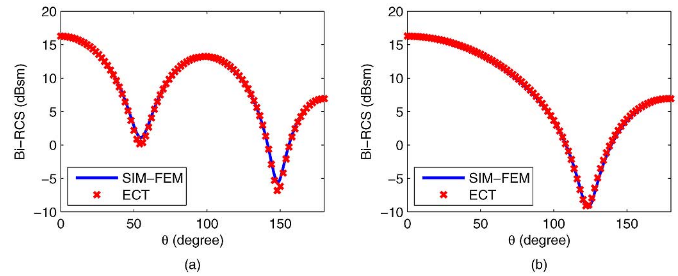

Fig. 3. RCS of a PEC cube of side length 1 m coated by a dielectric material of thickness 0.25 m on each side. (a) RCS versus θ for $\phi = 0^\circ$ and (b) $\phi = 90^\circ$. The 0.1-GHz plane wave is incident along the z direction and has x-polarization.

in Fig. 3 with the enlarged cell technique [11], [12] (a version of the conformal finite-difference time-domain method without reduction in time step increment). Excellent agreement has been observed.

IV. Conclusion

A 3-D spectral integral method (SIM) has been proposed as an alternative method for surface integral equations on cuboid objects. This method significantly reduces the computation time and memory requirements over the traditional method of moments. It can be used as a highly efficient radiation boundary condition for the finite-element method. Numerical results confirm that the method requires only $O(N^{1.5})$ memory and $O(N^{1.5}\log N)$ CPU time per iteration without making any approximation other than numerical discretization.

References

[1] R. F. Harrington, *Field Computation by Moment Methods*. New York: MacMillian, 1968.
[2] R. Coifman, V. Rokhlin, and S. Wandzura, "The fast multipole method for the wave equation: A pedestrian prescription," *IEEE Antennas Propagat. Mag.*, vol. 35, pp. 7–12, 1993.
[3] J. M. Song and W. C. Chew, "Multilevel fast-multipole algorithm for solving combined field integral equations of electromagnetic scattering," *Microw. Opt. Technol. Lett.*, vol. 10, pp. 15–19, 1995.
[4] E. Bleszynski, M. Bleszynski, and J. Jaroszewicz, "AIM: Adaptive integral method for solving large-scale electromagnetic scattering and radiation problems," *Radio Sci.*, vol. 31, pp. 1225–1251, 1996.
[5] M. F. Catedra, R. P. Torres, J. Basterrechea, and E. Gago, *The CG-FFT Method: Application of Signal Processing Techniques to Electromagnetics*. Boston, MA: Artech, 1995.
[6] J. M. Jin, J. L. Volakis, and J. D. Collins, "A finite element boundary integral method for scattering and radiation by two- and three-dimensional structures," *IEEE Antennas Propagat. Mag.*, vol. 33, pp. 22–32, Jun. 1991.
[7] J. L. Volakis, A. Chatterjee, and L. C. Kempel, "Review of the finite-element method for three-dimensional electromagnetic scattering," *J. Opt. Soc. Amer. A*, vol. 11, no. 4, pp. 1422–1433, Apr. 1994.
[8] J. M. Jin and V. V. Liepa, "Application of hybrid finite element method to electromagnetic scattering from coated cylinders," *IEEE Trans. Antennas Propagat.*, vol. AP-36, no. 1, pp. 50–54, Jan. 1988.
[9] J. Liu, Y. Lin, J.-H. Lee, E. Simsek, and Q. H. Liu, "Application of the hybrid spectral integral method with spectral element method," in *Proc. IEEE Ant. Propagat. Soc. Int. Symp.*, Honolulu, HI, Jun. 2007, p. 5611.
[10] E. Lucente and A. Monorchio, "A parallel iteration-free MOM algorithm based on the characteristic basis functions method," in *URSI Electromagnetic Theory Symposium*, Ottawa, ON, Canada, Jul. 26–28, 2007, pp. 1–3.
[11] T. Xiao and Q. H. Liu, "Enlarged cells for the conformal FDTD method to avoid the time step reduction," *IEEE Microw. Wireless Compon. Lett.*, vol. 14, no. 12, pp. 551–553, Dec. 2004.
[12] T. Xiao and Q. H. Liu, "A 3-D enlarged cell technique (ECT) for the conformal FDTD method," *IEEE Trans. Antennas Propagat.*, vol. 56, no. 3, pp. 765–773, Mar. 2008.