A Finite-Differences Based Numerical Solver To Find The Resonant Modes of Electromagnetic Waves in Dielectric Rings

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This work provides the coupled wave equations in cylindrical coordinates to study electromagnetic wave propagation in dielectric rings. Inspired by the shooting method, a two-step algorithm is proposed to solve these coupled nonlinear equations using finite differences. The accuracy of the formulation is validated through comparisons with numerical results obtained from commercial full-wave solvers for various geometries.

***Index Terms*—Finite difference methods, optical waveguides, ring resonators.**

I. Introduction

DIELECTRIC RINGS are key components in integrated photonics, enabling various applications such as optical filters and sensors [1]–[3]. Their compact size and resonant modes make them ideal for high-performance photonic circuits. Accurately determining their effective refractive index and resonant modes is crucial for optimizing light confinement, minimizing losses, and enhancing efficiency.

Current methods, like finite element and finite-difference time-domain simulations, implemented in solvers such as COMSOL and Tidy3D, offer accurate results. However, these general-purpose solvers do not fully exploit the resonance condition of dielectric rings, where the accumulated phase must equal an integer multiple of 2π for constructive interference.

Due to the lack of explicit expressions for electromagnetic waves confined in dielectric rings, we recently derived the coupled wave equations in cylindrical coordinates for such rings [4]. The resulting nonlinear equations are solved numerically using finite differences and a two-step algorithm inspired by the shooting method [5], [6]. We first use an approximate azimuthal mode number determined with the wavelength, ring radius, and refractive index to solve the eigenvalue problem. Then, we refine this estimate and resolve the problem to meet the phase continuity condition. The method yields electric and magnetic field profiles, validated through several examples. Our results agree closely with commercial solvers, confirming the method's accuracy.

II. Numerical Formulation

Figure 1 (a) illustrates a dielectric ring with a central radius of R_c placed in a background with a cylindrical symmetry with respect to the z-axis. When they are in resonance, the electromagnetic waves confined in the ring should have an $e^{-jm\phi}$ dependence, where $m = \beta R_c$ is an integer representing the azimuthal mode order and β is the propagation constant. Due to the cylindrical symmetry, the problem can be treated in two dimensions as shown in Fig. 1 (b). We use following expressions to describe the electric ($\mathbf{E}$) and magnetic ($\mathbf{H}$) fields

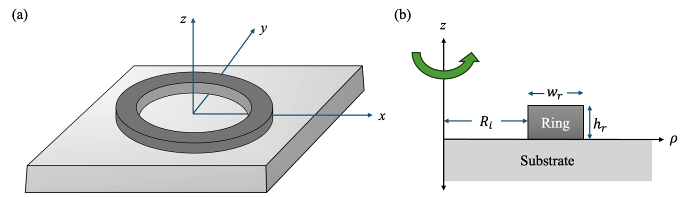

Fig. 1. A dielectric ring with a central radius of R_c is placed on a substrate. The ring's inner radius, width, and height are R_i, w_r and h_r, where $R_c = R_i + w_r/2$. (a) and (b) are three- and two-dimensional views.

$$\mathbf{E}(\rho,\phi,z) = \left\{\hat{\rho}E_\rho(\rho,z) + \hat{\phi}E_\phi(\rho,z) + \hat{z}E_z(\rho,z)\right\} e^{-jm\phi}, \tag{1}$$

$$\mathbf{H}(\rho,\phi,z) = \left\{\hat{\rho}H_\rho(\rho,z) + \hat{\phi}H_\phi(\rho,z) + \hat{z}H_z(\rho,z)\right\} e^{-jm\phi}. \tag{2}$$

We start with Maxwell's equations for the electric and magnetic fields in a source-free, lossless, and non-magnetic medium and obtain the following wave equations

$$\nabla^2\mathbf{E} + \nabla\left(\frac{1}{\varepsilon_r}\nabla\varepsilon_r \cdot \mathbf{E}\right) + k_0^2\varepsilon_r\mathbf{E} = 0, \tag{3}$$

$$-\varepsilon_r\nabla\left(\frac{1}{\varepsilon_r}\right) \times \nabla \times \mathbf{H} + \nabla^2\mathbf{H} + k_0^2\varepsilon_r\mathbf{H} = 0. \tag{4}$$

where ε_r is the relative electrical permittivity and μ_0 is the magnetic permeability of vacuum.

We first expand the electric field wave vector equation by inserting the electric field expression, Eq. (1), and using vector identities for the cylindrical coordinate system. We obtain three coupled equations for the fields along the $\hat{\rho}$, $\hat{\phi}$, and $\hat{z}$ directions. Then, we do the same for the magnetic field wave vector equation. In the final step, we obtain the mixed formulation, which allows for direct enforcement of boundary conditions on both electric and magnetic fields at interfaces between materials [7], by expressing H_ϕ in terms of E_z and E_ρ and by expressing E_ϕ in terms of H_z and H_ρ. The final set of equations is as follows

$$\left(\mathcal{L} - \frac{1}{R_c^2} + \frac{\rho^2}{R_c^2}\frac{\partial}{\partial \rho}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial \rho}\right) E_\rho + \left(\frac{\rho^2}{R_c^2}\frac{\partial}{\partial \rho}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial z}\right) E_z + \frac{2m}{\omega \epsilon R_c^2}\left(\frac{\partial H_\rho}{\partial z} - \frac{\partial H_z}{\partial \rho}\right) = \beta^2 E_\rho, \tag{5}$$

$$\frac{\rho^2}{R_c^2}\frac{\partial}{\partial z}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial \rho} E_\rho + \mathcal{L} E_z + \frac{\rho^2}{R_c^2}\frac{\partial}{\partial z}\frac{1}{\varepsilon}\frac{\partial \varepsilon}{\partial z} E_z = \beta^2 E_z, \tag{6}$$

$$\left(\mathcal{L} - \frac{1}{R_c^2} - \frac{\rho^2}{R_c^2}\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial z}\frac{\partial}{\partial z}\right) H_\rho + \frac{\rho^2}{R_c^2}\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial z}\frac{\partial}{\partial z} H_z + \frac{2m}{\omega \mu_0 R_c^2}\left(\frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z}\right) = \beta^2 H_\rho, \tag{7}$$

$$\frac{\rho^2}{R_c^2}\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \rho}\frac{\partial H_\rho}{\partial z} + \left(\mathcal{L} - \frac{\rho^2}{R_c^2}\frac{1}{\epsilon}\frac{\partial \epsilon}{\partial \rho}\frac{\partial}{\partial \rho}\right) H_z = \beta^2 H_z, \tag{8}$$

where

$$\mathcal{L} = \frac{\rho^2}{R_c^2}\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k_0^2 \varepsilon_r\right). \tag{9}$$

In the above, we move all the terms with m^2 to the right sides, replace those m^2s with $\beta^2 R_c^2$'s, and multiply both sides of all equations with ρ^2/R_c^2. Eqs. (5)–(8) can be cast into a matrix equation such as

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ M_5 & M_6 & M_7 & M_8 \\ M_9 & M_{10} & M_{11} & M_{12} \\ M_{13} & M_{14} & M_{15} & M_{16} \end{bmatrix} \begin{bmatrix} E_\rho \\ E_z \\ H_\rho \\ H_z \end{bmatrix} = \beta^2 \begin{bmatrix} E_\rho \\ E_z \\ H_\rho \\ H_z \end{bmatrix}, \tag{10}$$

and solved numerically with finite differences for a given m value.

The problem in this formulation is that since $m = \beta R_c$, we have the unknown, "β", on both sides of Eq. (10). In a similar fashion to the "shooting methods" [5], we use an approximate $\beta_{\text{approx}} = \xi k_0 n_{\text{r}}$, where ξ is an arbitrary positive number closer to 1.0 and n_{r} is the refractive index of the ring, to calculate the resonant wave number β. Then, we build the Hamiltonian matrix with this new approximate propagation constant, determine the eigenvalues, and repeat this process until the difference between the previous and current propagation constants is negligibly small ($< 10^{-8}$).

The computational domain is discretized on a rectangular grid by taking N_ρ evenly spaced points along the ρ-direction and N_z evenly spaced points along the z-direction. First- and second-order derivatives are calculated using central finite differences, applying Neumann boundary conditions to enable a smaller computational domain, as Dirichlet conditions would require a larger domain by forcing the fields to be zero at the boundaries.

III. Numerical Results

Assume we have a dielectric ring with $n_{\text{r}} = 2$, $R_c = 200$ μm, $w_r = 1$ μm, and $h_r = 0.5$ μm, placed on top of a substrate with a refractive index of 1.5. The wavelength of the excitation is 1550 nm. For the numerical solution, we use a 200×100 grid on the $\rho - z$ plane, where the mesh sampling density along both directions is 25 nm. The effective indices of the first two resonant modes are calculated with our formulation to be 1.6265 and 1.5478. A commercial full wave solver, COMSOL Multiphysics, determines the values of 1.6302 and 1.5495. The difference between them is less than 0.3 % for both modes. Figure 2 shows the field profiles for the first resonant mode. At the conference, we will provide several other examples to demonstrate the accuracy and flexibility of the formulation.

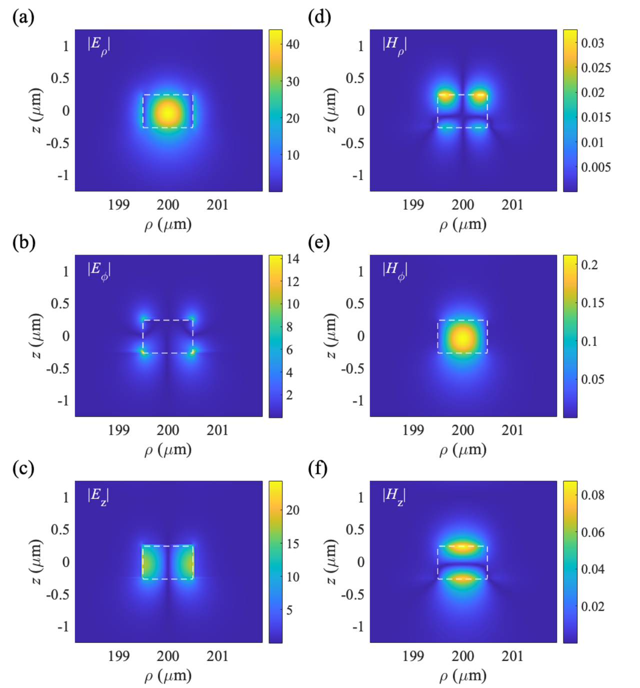

Fig. 2. The field profiles for the first resonant mode: ρ, ϕ, and z components of the (a)–(c) electric field and (d)–(f) magnetic field intensity.

IV. Conclusion

A complete set of coupled wave equations for the electromagnetic waves confined in dielectric rings is provided. These equations are then discretized using finite differences and solved as an eigenvalue problem. Our approach offers a computationally efficient alternative to full-wave electromagnetic solvers, particularly for large-scale simulations. The results obtained from our method exhibit good agreement with those from commercial full-wave electromagnetic solvers, validating the accuracy of the formulation.

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