

Finite-Differences-Based Solvers for Wave Propagation in Dielectric Waveguides and Rings

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Abstract—Dielectric waveguides and ring resonators are foundational structures in photonics, enabling applications such as optical frequency comb generation, filtering, and sensing. This work presents a complete formulation to analyze electromagnetic wave propagation in these structures, incorporating the shooting method to accurately calculate propagation constants, effective indices, and modal fields. With the help of the coupled-mode theory, we investigate the transmission spectrum of a Si_3N_4 waveguide coupled to a Si_3N_4 ring under the critical resonance condition. Our numerical results agree with the ones generated with COMSOL Multiphysics.

I. INTRODUCTION

Dielectric waveguides coupled to ring resonators, as shown in Fig. 1, are crucial structures in photonics to manipulate and confine light with high precision, enabling diverse applications such as optical frequency comb generation, filtering, and sensing [1]. These structures rely on the interplay between waveguide and resonator modes, which govern how electromagnetic waves propagate and interact. Modal analysis is crucial to understanding the supported modes, their effective indices, and loss characteristics, as these factors directly influence the device's performance, such as resonance quality (Q -factor), spectral selectivity, and coupling efficiency [2]. In this work, we first provide compact formulations to calculate the modes of electromagnetic waves propagating in dielectric waveguides and rings. Since the final equations are non-linear, we use the shooting method [3], [4] to find the propagation constants. Then, we use coupled-mode theory [2] for analyzing energy transfer between the waveguide and resonator, capturing the resonance conditions and modal interactions critical for optical frequency comb and filter design.

II. FORMULATION

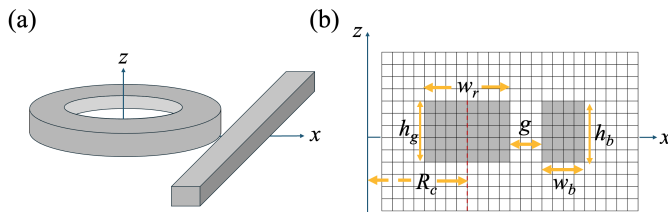


Fig. 1. A dielectric bus waveguide next to a dielectric ring. The ring is centered at $\rho = 0$. R_c , w_r , and h_g are the central radius, width, and height of the ring. g is the gap between the ring and waveguide. w_b and h_b are the waveguide's width and height. The waveguide is excited with an electromagnetic wave at the wavelength of λ . The computation domain is meshed uniformly along the ρ and z axes. The mesh density is set to $\lambda/20$.

A. Electromagnetic Waves in Dielectric Waveguides

We assume that the waveguide is infinitely long along the y -axis, and we define the electric field

$$\mathbf{E}(x, y, z) = \sum_{\Psi} \hat{\Psi} E_{\Psi}(x, z) e^{-j\beta_b y}, \quad (1)$$

where Ψ is x , y , or z , and β_b is the propagation constant along the y -axis. Substituting this into the wave equation for the electric field, we obtain the following three sub-equations:

$$\mathcal{L}_w\{E_x\} + \frac{\partial}{\partial x} \frac{1}{\varepsilon} \left(\frac{\partial \varepsilon}{\partial x} E_x + \frac{\partial \varepsilon}{\partial z} E_z \right) = \beta_b^2 E_x, \quad (2)$$

$$\mathcal{L}_w\{E_y\} - j\beta_b \frac{1}{\varepsilon} \left(\frac{\partial \varepsilon}{\partial x} E_x + \frac{\partial \varepsilon}{\partial z} E_z \right) = \beta_b^2 E_y, \quad (3)$$

$$\mathcal{L}_w\{E_z\} + \frac{\partial}{\partial z} \frac{1}{\varepsilon} \left(\frac{\partial \varepsilon}{\partial x} E_x + \frac{\partial \varepsilon}{\partial z} E_z \right) = \beta_b^2 E_z, \quad (4)$$

where ε is a function of position, and

$$\mathcal{L}_w\{E_t\} = \frac{\partial^2 E_t}{\partial x^2} + \frac{\partial^2 E_t}{\partial y^2} + k_0^2 \varepsilon_r E_t. \quad (5)$$

Then, we cast these equations into the following matrix equation

$$\begin{bmatrix} M_1 & M_2 & M_3 \\ M_4 & M_5 & M_6 \\ M_7 & M_8 & M_9 \end{bmatrix} \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix} = \beta_b^2 \begin{bmatrix} E_x \\ E_y \\ E_z \end{bmatrix}. \quad (6)$$

B. Electromagnetic Waves in Dielectric Rings

The electromagnetic waves confined in a dielectric ring with a central radius of R_c should have an $e^{-jm\phi}$ dependence, where $m = \beta R_c$ is an integer representing the azimuthal mode order and β is the propagation constant. The following expression is used to describe fields inside the ring

$$\mathbf{A}(\rho, \phi, z) = \left\{ \hat{\rho} A_{\rho}(\rho, z) + \hat{\phi} A_{\phi}(\rho, z) + \hat{z} A_z(\rho, z) \right\} e^{-jm\phi}, \quad (7)$$

where $\mathbf{A}$ is either electric ($\mathbf{E}$) or magnetic ($\mathbf{H}$) field. We derive the wave equations in cylindrical coordinates from Maxwell's equations. Then we express H_{ϕ} in terms of E_z and E_{ρ} and express E_{ϕ} in terms of H_z and H_{ρ} and obtain the following set of equations:

$$\left(\mathcal{L} - \frac{1}{R_c^2} + \frac{\rho^2}{R_c^2} \frac{\partial}{\partial \rho} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} \right) E_{\rho} + \left(\frac{\rho^2}{R_c^2} \frac{\partial}{\partial \rho} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} \right) E_z + \frac{2m}{\omega \varepsilon R_c^2} \left(\frac{\partial H_{\rho}}{\partial z} - \frac{\partial H_z}{\partial \rho} \right) = \beta^2 E_{\rho}, \quad (8)$$

$$\frac{\rho^2}{R_c^2} \frac{\partial}{\partial z} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} E_{\rho} + \mathcal{L} E_z + \frac{\rho^2}{R_c^2} \frac{\partial}{\partial z} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} E_z = \beta^2 E_z, \quad (9)$$

$$\left(\mathcal{L} - \frac{1}{R_c^2} - \frac{\rho^2}{R_c^2} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} \frac{\partial}{\partial z}\right) H_\rho + \frac{\rho^2}{R_c^2} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial z} \frac{\partial}{\partial z} H_z + \frac{2m}{\omega \mu_0 R_c^2} \left(\frac{\partial E_z}{\partial \rho} - \frac{\partial E_\rho}{\partial z}\right) = \beta^2 H_\rho, \quad (10)$$

$$\frac{\rho^2}{R_c^2} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} \frac{\partial}{\partial z} H_\rho + \left(\mathcal{L} - \frac{\rho^2}{R_c^2} \frac{1}{\varepsilon} \frac{\partial \varepsilon}{\partial \rho} \frac{\partial}{\partial \rho}\right) H_z = \beta^2 H_z, \quad (11)$$

where

$$\mathcal{L} = \frac{\rho^2}{R_c^2} \left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho} \frac{\partial}{\partial \rho} + \frac{\partial^2}{\partial z^2} + k_0^2 \varepsilon_r \right). \quad (12)$$

Above, we move all the terms with m^2 to the right sides, replace those m^2 s with $\beta^2 R_c^2$'s, and multiply both sides of all equations with ρ^2/R_c^2 . Eqs. (8)–(11) can be cast into a matrix equation such as

$$\begin{bmatrix} M_1 & M_2 & M_3 & M_4 \\ M_5 & M_6 & M_7 & M_8 \\ M_9 & M_{10} & M_{11} & M_{12} \\ M_{13} & M_{14} & M_{15} & M_{16} \end{bmatrix} \begin{bmatrix} E_\rho \\ E_z \\ H_\rho \\ H_z \end{bmatrix} = \beta^2 \begin{bmatrix} E_\rho \\ E_z \\ H_\rho \\ H_z \end{bmatrix}, \quad (13)$$

and solved numerically with the finite differences for a given m value.

C. Shooting Method

Since we have the unknown propagation constant on the left side of Eqs. (4), (8) and, (10), we first use an approximate value to solve the Eqs. (6) and (13), and we compute the eigenvalues β_b [3] and β [4]. Then, we use these new propagation constants and solve the matrix equations again. We repeat this process till the differences between the previous and current values are negligibly small.

III. NUMERICAL RESULTS

We assume that the waveguide and ring are made from Si_3N_4 . The surrounding material is SiO_2 . $\lambda = 1.55 \mu\text{m}$, $R_c = 80 \mu\text{m}$, $w_r = w_b = 1.4 \mu\text{m}$, $h_r = h_b = 0.8 \mu\text{m}$, and $g = 0.6 \mu\text{m}$. In Fig. 2, we plot the electric field intensity inside the dielectric waveguide and the ring for the first resonant mode calculated with this formulation (first two rows) and COMSOL (last two rows). Our solver and COMSOL determine the effective index of the waveguide and ring to be 1.8332 and 1.8342. We calculate the group index of the ring to be 2.1352. Then, we use the coupled-mode theory [2] to understand the non-linear nature of coupling and frequency comb generation. The coupling quality factor [5] is computed to be 6.4 million. As shown in Fig. 3, the free-spectral range (FSR) and full-width at half maximum are calculated to be 279 GHz and 62 MHz. At the conference, we will compare our numerical results with experimental data from the literature.

IV. CONCLUSION

In this work, we presented a detailed formulation for analyzing electromagnetic wave propagation in dielectric waveguides and rings. Employing the shooting method, we accurately calculated the propagation constants, effective indices, and modal fields inside these structures. Further analysis conducted with

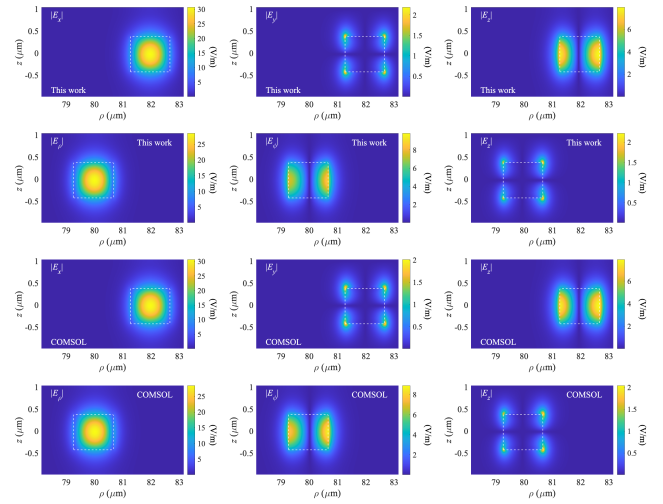


Fig. 2. Electric field intensity of the first resonant modes inside the waveguide (first and third rows) and ring (second and fourth rows) determined by this work and COMSOL, respectively.

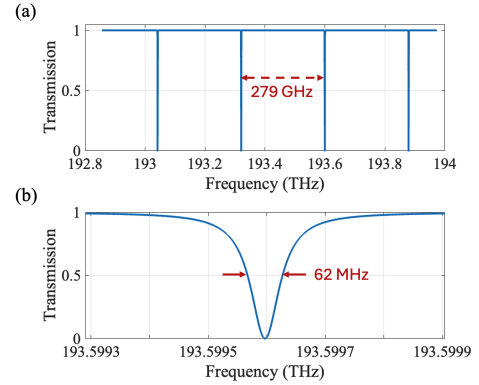


Fig. 3. Transmission through a Si_3N_4 waveguide coupled to a Si_3N_4 ring resonator as a function of frequency: (a) over 4 FSR, and (b) zoom-in at one of the dips.

the coupled-mode theory yielded the transmission spectrum through a dielectric waveguide coupled to a dielectric ring resonator, confirming the optical frequency comb generation.

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