SECTION 2.3: LONG AND SYNTHETIC POLYNOMIAL DIVISION

PART A: LONG DIVISION

Ancient Example with Integers

$$4\overline{)\,9\,}$$ with quotient 2

$$\begin{array}{r} 2 \\ 4\overline{)\ 9} \\ -8 \\ \hline 1 \end{array}$$

We can say: $\dfrac{9}{4} = 2 + \dfrac{1}{4}$

By multiplying both sides by 4, this can be rewritten as:

$$9 = 4 \cdot 2 + 1$$

In general: $\dfrac{\text{dividend}, f}{\text{divisor}, d} = \left(\text{quotient}, q\right) + \dfrac{\left(\text{remainder}, r\right)}{d}$

where either:

$r = 0$ (in which case d <u>divides evenly</u> into f), or

$\dfrac{r}{d}$ is a positive <u>proper</u> fraction: i.e., $0 < r < d$

<u>Technical Note</u>: We assume f and d are positive integers, and q and r are nonnegative integers.

<u>Technical Note</u>: We typically assume $\dfrac{f}{d}$ is <u>improper</u>: i.e., $f \ge d$. Otherwise, there is no point in dividing this way.

<u>Technical Note</u>: Given f and d, q and r are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by d, $\dfrac{f}{d} = q + \dfrac{r}{d}$ can be rewritten as:

$$f = d \cdot q + r$$

Now, we will perform polynomial division on $\dfrac{f(x)}{d(x)}$ so that we get:

$$\frac{f(x)}{d(x)} = q(x) + \frac{r(x)}{d(x)}$$

where either:

$r(x) = 0$, in which case $d(x)$ <u>divides evenly</u> into $f(x)$, or

$\dfrac{r(x)}{d(x)}$ is a <u>proper</u> rational expression: i.e., $\deg\big(r(x)\big) < \deg\big(d(x)\big)$

<u>Technical Note</u>: We assume $f(x)$ and $d(x)$ are nonzero polynomials, and $q(x)$ and $r(x)$ are polynomials.

<u>Technical Note</u>: We assume $\dfrac{f(x)}{d(x)}$ is <u>improper</u>; i.e., $\deg\big(f(x)\big) \geq \deg\big(d(x)\big)$.

Otherwise, there is no point in dividing.

<u>Technical Note</u>: Given $f(x)$ and $d(x)$, $q(x)$ and $r(x)$ are unique by the Division Algorithm (really, it's a theorem).

By multiplying both sides by $d(x)$, $\dfrac{f(x)}{d(x)} = q(x) + \dfrac{r(x)}{d(x)}$ can be rewritten as:

$$f(x) = d(x) \cdot q(x) + r(x)$$

Example

Use Long Division to divide: $\dfrac{-5+3x^2+6x^3}{1+3x^2}$

Solution

Warning: First, write the N and the D in descending powers of x.

Warning: Insert "missing term placeholders" in the N (and perhaps even the D) with "0" coefficients. This helps you avoid errors. We get:

$$\frac{6x^3+3x^2+0x-5}{3x^2+0x+1}$$

Let's begin the Long Division:

$$3x^2+0x+1\overline{)\,6x^3+3x^2+0x-5}$$

The steps are similar to those for $4\overline{)\,9}$.

Think: How many "times" does the leading term of the divisor ($3x^2$) "go into" the leading term of the dividend ($6x^3$)? We get: $\dfrac{6x^3}{3x^2}=2x$, which goes into the quotient.

$$\begin{array}{r} 2x \\ 3x^2+0x+1\overline{)\,6x^3+3x^2+0x-5} \end{array}$$

Multiply the $2x$ by the divisor and write the product on the next line.

Warning: Line up like terms to avoid confusion!

$$\begin{array}{r} 2x \\ 3x^2+0x+1\overline{)\,6x^3+3x^2+0x-5} \\ 6x^3+0x^2+2x \end{array}$$

Warning: We must **subtract** this product from the dividend. People have a much easier time adding than subtracting, so let's flip the sign on each term of the product, and add the result to the dividend. To avoid errors, we will cross out our product and do the sign flips on a separate line before adding.

Warning: Don't forget to bring down the -5.

$$
\begin{array}{r}
2x \\
3x^2 + 0x + 1 \overline{)\; 6x^3 + 3x^2 + 0x - 5} \\
\cancel{6x^3 + 0x^2 + 2x} \\
-6x^3 - 0x^2 - 2x \\
\hline
3x^2 - 2x - 5
\end{array}
$$

We now treat the expression in blue above as our new dividend. Repeat the process.

$$
\begin{array}{r}
2x + 1 \\
3x^2 + 0x + 1 \overline{)\; 6x^3 + 3x^2 + 0x - 5} \\
\cancel{6x^3 + 0x^2 + 2x} \\
-6x^3 - 0x^2 - 2x \\
\hline
3x^2 - 2x - 5
\end{array}
$$

$$
\begin{array}{r}
2x + 1 \\
3x^2 + 0x + 1 \overline{)\; 6x^3 + 3x^2 + 0x - 5} \\
\cancel{6x^3 + 0x^2 + 2x} \\
-6x^3 - 0x^2 - 2x \\
\hline
3x^2 - 2x - 5 \\
3x^2 + 0x + 1
\end{array}
$$

$$\begin{array}{r}
2x + 1 \\
3x^2 + 0x + 1{\overline{\smash{\big)}\,6x^3 + 3x^2 + 0x - 5}} \\
\underline{\,6x^3 + 0x^2 + 2x} \\
-6x^3 - 0x^2 - 2x \\
\underline{\,3x^2 - 2x - 5} \\
3x^2 + 0x + 1 \\
-3x^2 - 0x - 1 \\
\underline{} \\
-2x - 6
\end{array}$$

We can now stop the process, because the degree of the new dividend is less than the degree of the divisor. The degree of $-2x - 6$ is 1, which is less than the degree of $3x^2 + 0x + 1$, which is 2. This guarantees that the fraction in our answer is a **proper** rational expression.

Our answer is of the form: $q(x) + \dfrac{r(x)}{d(x)}$

$$2x + 1 \ + \ \frac{-2x - 6}{3x^2 + 1}$$

If the leading coefficient of $r(x)$ is negative, then we factor a -1 out of it.

$$\underline{\text{Answer:}} \quad \mathbf{2x + 1 \ - \ \frac{2x + 6}{3x^2 + 1}}$$

<u>Warning</u>: Remember to flip **every** sign in the numerator.

<u>Warning</u>: If the N and the D of our fraction have any common factors aside from ± 1, they must be canceled out. Our fraction here is simplified as is.

PART B: SYNTHETIC DIVISION

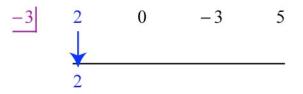There's a great short cut if the divisor is of the form $x - k$.

<u>Example</u>

Use Synthetic Division to divide: $\dfrac{2x^3 - 3x + 5}{x + 3}$.

<u>Solution</u>

The divisor is $x + 3$, so $k = -3$.
<u>Think</u>: $x + 3 = x - (-3)$.
We will put -3 in a half-box in the upper left of the table below.

Make sure the N is written in standard form.
Write the coefficients in order along the first row of the table.
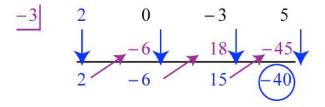Write a "placeholder 0" if a term is missing.

Bring down the first coefficient, the "2."

$$\begin{array}{c|cccc}
-3 & 2 & 0 & -3 & 5 \\
& \downarrow & & & \\
\hline
& 2 & & & \\
\end{array}$$

The $\downarrow$ arrow tells us to add down the column and write the sum in the third row.

The $\nearrow$ arrow tells us to multiply the blue number by k (here, -3) and write the product one column to the right in the second row.

Circle the lower right number.

$$\begin{array}{r|rrrr} -3 & 2 & 0 & -3 & 5 \\ & & -6 & 18 & -45 \\ \hline & 2 & -6 & 15 & \boxed{-40} \end{array}$$

Since we are dividing a 3rd-degree dividend by a 1st-degree divisor, our answer begins with a 2nd-degree term.

The third (blue) row gives the coefficients of our quotient in descending powers of x. The circled number is our remainder, which we put over our divisor and factor out a -1 if appropriate.

<u>Note</u>: The remainder **must** be a **constant**, because the divisor is linear.

<u>Answer</u>: $2x^2 - 6x + 15 - \dfrac{40}{x+3}$

<u>Related Example</u>

Express $f(x) = 2x^3 - 3x + 5$ in the following form:

$f(x) = d(x) \cdot q(x) + r$, where the divisor $d(x) = x + 3$.

<u>Solution</u>

We can work from our previous Answer. Multiply both sides by the divisor:

$$\frac{2x^3 - 3x + 5}{x+3} = 2x^2 - 6x + 15 - \frac{40}{x+3}$$
$$2x^3 - 3x + 5 = (x+3) \cdot (2x^2 - 6x + 15) - 40$$

<u>Note</u>: Synthetic Division works even if $k = 0$. What happens?

PART C: REMAINDER THEOREM

Remainder Theorem

If we are dividing a polynomial $f(x)$ by $x - k$, and if r is the remainder, then $f(k) = r$.

In our previous Examples, we get the following fact as a bonus.

$$f(-3) = \boxed{-40}$$

Synthetic Division therefore provides an efficient means of evaluating polynomial functions. (It may be much better than straight calculator button-pushing when dealing with polynomials of high degree.) We could have done the work in Part B if we had wanted to evaluate $f(-3)$, where $f(x) = 2x^3 - 3x + 5$.

Warning: Do **not** flip the sign of -3 when writing it in the half-box. People get the "sign flip" idea when they work with polynomial division.

Technical Note: See the short Proof on p.192.

PART D: ZEROS, FACTORING, AND DIVISION

Recall from Section 2.2:

Factor Theorem

If $f(x)$ is a nonzero polynomial and k is a real number, then

k is a zero of $f \iff (x-k)$ is a factor of $f(x)$.

Technical Note: The Proof on p.192 uses the Remainder Theorem to prove this.

What happens if either Long or Synthetic polynomial division gives us a 0 remainder? Then, we can at least partially factor $f(x)$.
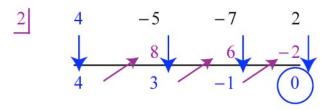
Example

Show that 2 is a zero of $f(x) = 4x^3 - 5x^2 - 7x + 2$.

Note: We saw this $f(x)$ in Section 2.2.

Note: In Section 2.5, we will discuss a trick for finding such a zero.

Factor $f(x)$ completely, and find **all** of its real zeros.

Solution

We will use Synthetic Division to show that 2 is a zero:

By the Remainder Theorem, $f(2) = 0$, and so 2 is a zero.

By the Factor Theorem, $(x-2)$ must be a factor of $f(x)$.

Technical Note: This can be seen from the form
$f(x) = d(x) \cdot q(x) + r$. Since $r = 0$ when $d(x) = x - 2$, we have:

$$f(x) = (x-2) \cdot q(x),$$

where $q(x)$ is some (here, quadratic) polynomial.

We can find $q(x)$, the other (quadratic) factor, by using the last row of the table.

$$f(x) = (x-2) \cdot \left(4x^2 + 3x - 1\right)$$

Factor $q(x)$ completely over the reals:

$$f(x) = (x-2)(4x-1)(x+1)$$

The zeros of $f(x)$ are the zeros of these factors:

$$2, \frac{1}{4}, -1$$

Below is a graph of $f(x) = 4x^3 - 5x^2 - 7x + 2$. Where are the x-intercepts?