## .3 THE ASSOCIATED MIXED-RADIX SYSTEM

Magnitude comparison, sign detection, and overflow detection for the residue number system can be facilitated by converting the given residue representations into the associated mixed-radix number system. This is a weighted number system, with the representation for a number $X$ given by

$$X = a_N \cdot (m_{N-1} \cdot m_{N-2} \cdots m_1) + \cdots + a_3 \cdot (m_2 \cdot m_1) + a_2 \cdot m_1 + a_1 \quad (11.10)$$

with the digits $a_i$ satisfying

$$0 \leq a_i < m_i; \qquad i = 1, 2, \cdots, N. \quad (11.11)$$

Being a weighted number system implies that magnitude comparison is straightforward. For example, the values 0,1,2,3,4 and 5 in the mixed-radix system associated with the (3,2) residue system (see Table 11.1) are represented by (0,0), (0,1), (1,0), (1,1), (2,0), and (2,1), respectively. The value of a pair $(a_2, a_1)$ in this mixed-radix system is $2 \cdot a_2 + a_1$.

**Example 11.4**

In the mixed-radix system associated with the $(m_4, m_3, m_2, m_1) = (7, 5, 3, 2)$ residue system, a number $X$ is represented by $(a_4, a_3, a_2, a_1)$, where

$$X = 30 \cdot a_4 + 6 \cdot a_3 + 2 \cdot a_2 + a_1$$

and the digits $a_i$ satisfy $0 \leq a_4 < 7$, $0 \leq a_3 < 5$, $0 \leq a_2 < 3$ and $0 \leq a_1 < 2$. The numbers 43 and 37 are represented by (1,3,1,1) and (2,2,1,1) in the given residue system, respectively. The corresponding representations in the associated mixed-radix system are (1,2,0,1) and (1,1,0,1), respectively. These last two representations can be compared indicating that 43 is greater than 37. □

Any two numbers in a given residue system can be compared by converting them into the associated mixed-radix system. Converting a number represented by $(x_N, x_{N-1}, \cdots, x_1)$ in the residue system to the associated mixed-radix representation $(a_N, a_{N-1}, \cdots, a_1)$ is performed using the following equations [7]:

$$\begin{aligned}
a_1 &= X \bmod m_1 = x_1 \\
a_2 &= (X - a_1)\left|\frac{1}{m_1}\right| \bmod m_2 \qquad (11.12) \\
a_3 &= \left((X - a_1)\left|\frac{1}{m_1}\right| - a_2\right)\left|\frac{1}{m_2}\right| \bmod m_3 \\
&\vdots
\end{aligned}$$

This calculation can be done in residue arithmetic, as can be easily verified through the following representation of the procedure in Equation (11.12):

$$\begin{aligned} Y_{i+1} &= (Y_i - a_i)\,|\frac{1}{m_i}| \quad \text{with } Y_1 = X \\ a_i &= Y_i \bmod m_i \end{aligned} \tag{11.13}$$

**Example 11.5**

To convert a number $X$ represented by $(x_4, x_3, x_2, x_1)$ in the residue system with the moduli $(m_4, m_3, m_2, m_1) = (7, 5, 3, 2)$ to the associated mixed-radix system, the following equations can be used:

$$\begin{aligned} a_1 &= X \bmod 2 = x_1, \\ a_2 &= (X - a_1)|\frac{1}{2}| \bmod 3, \\ a_3 &= \left((X - a_1)|\frac{1}{2}| - a_2\right)|\frac{1}{3}| \bmod 5, \\ a_4 &= \left(\left((X - a_1)|\frac{1}{2}| - a_2\right)|\frac{1}{3}| - a_3\right)|\frac{1}{5}| \bmod 7. \end{aligned}$$

It is more convenient to follow the algorithm in Equation (11.13) and execute the conversion in the residue system. For example, we convert the number 43 represented by (1,3,1,1) as follows:

$$Y_1 = (1, 3, 1, 1) \quad \text{and therefore,} \quad a_1 = Y_1 \bmod 2 = x_1 = 1.$$

To obtain $Y_2$ we first subtract $a_1$ from $Y_1$, yielding $(0, 2, 0, -)$. Note that only the first three digits in $Y_2$ are of interest, since $a_1$ is already known. We then multiply by $|\frac{1}{2}|$, which equals $(4, 3, 2, -)$, obtaining $Y_2 = (0, 1, 0, -)$. Thus, $a_2 = Y_2 \bmod 3 = 0$. Subtracting $a_2 = 0$ yields $(0, 1, -, -)$. Next we multiply by $|\frac{1}{3}|$, which equals $(5, 2, -, -)$, yielding $Y_3 = (0, 2, -, -)$. Therefore, $a_3 = Y_3 \bmod 5 = 2$. Subtracting $a_3 = 2$ we get $(5, -, -, -)$. We then multiply by $|\frac{1}{5}|_7 = 3$, yielding $Y_4 = (1, -, -, -)$. Thus, $a_4 = 1$ and the representation of 43 in the mixed-radix system is $(a_4, a_3, a_2, a_1) = (1, 2, 0, 1)$. □

The mixed-radix system is useful for overflow detection as well. For this purpose, we should add a redundant modulus $m_{N+1}$ to the basic set of $N$ moduli. Here, the term *redundant modulus* means that we use only the range determined by the original $N$ moduli. For overflow detection we convert the given representation $(x_{N+1}, x_N, \cdots, x_1)$ to the associated mixed-radix system. If $a_{N+1} \neq 0$ then an overflow has occurred.