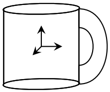
3D Transformations

CMSC 435/634

Transformation

Webster: The operation of changing one configuration or expression into another in accordance with a mathematical rule



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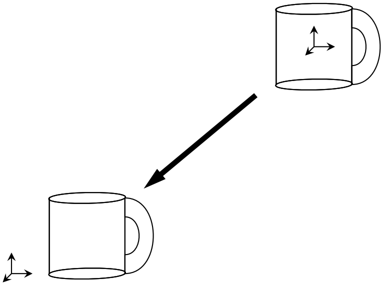

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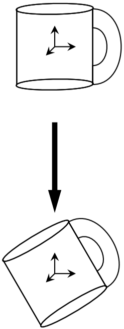

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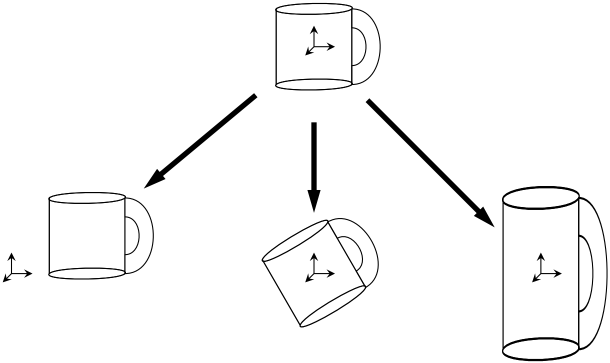
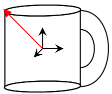


Using Transformation

- Points on object represented as vector offset from origin



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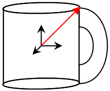

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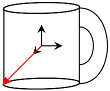
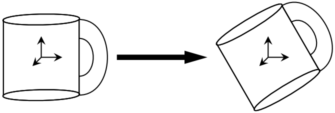


Using Transformation

- Points on object represented as vector offset from origin
- Transform is a vector to vector function
  - $\vec{p}' = f(\vec{p})$



Using Transformation

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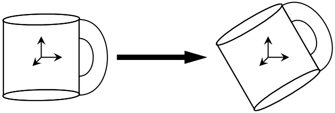
  - $\vec{p}' = f(\vec{p})$
- Relativity:
  - From $\vec{p}'$ point of view, object is transformed



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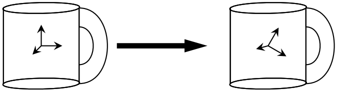

Using Transformation

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  - $\vec{p}' = f(\vec{p})$
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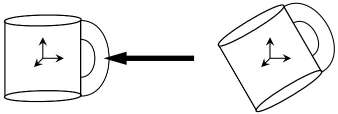
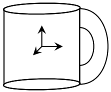
  - From $\vec{p}$ point of view, coordinate system changes
- Inverse transform, $\vec{p} = f^{-1}(\vec{p}')$



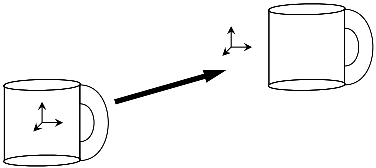
Composing Transforms

- Order matters



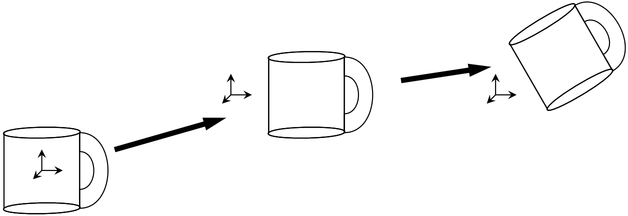
Composing Transforms

- Order matters
  - $T(\vec{p})$



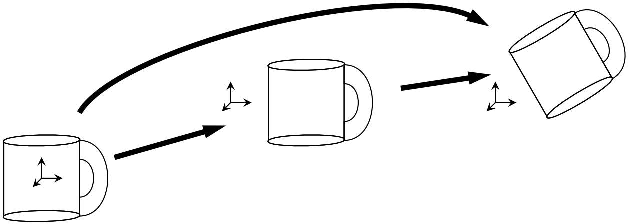
Composing Transforms

- Order matters
  - $R(T(\vec{p}))$



Composing Transforms

- Order matters
  - $R(T(\vec{p})) = R \circ T(\vec{p})$



Composing Transforms

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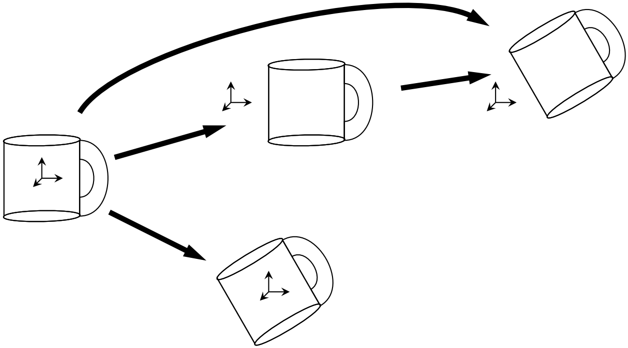

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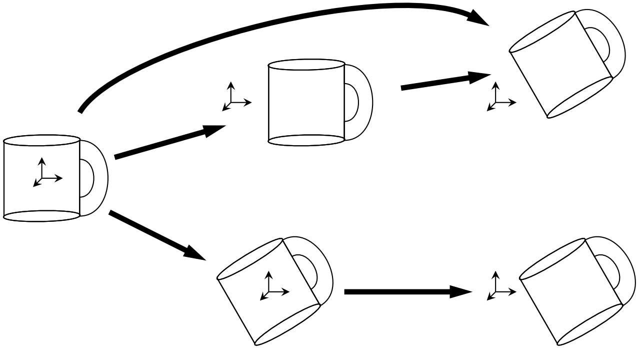

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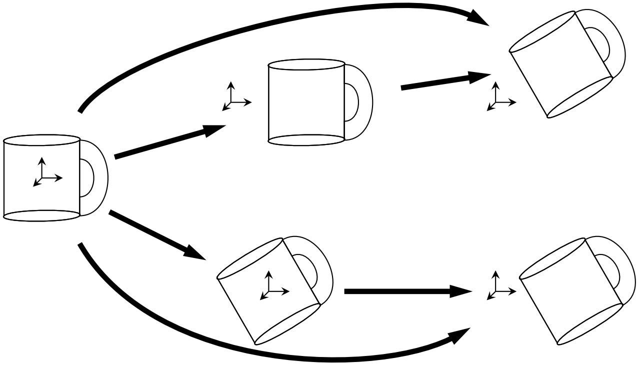
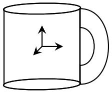


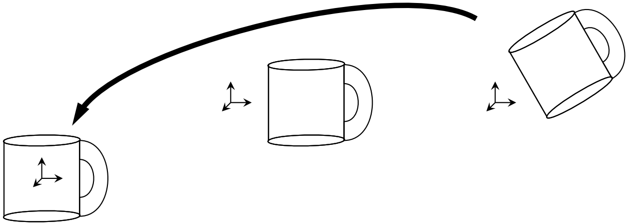
Inverting Composed Transforms

- Reverse order



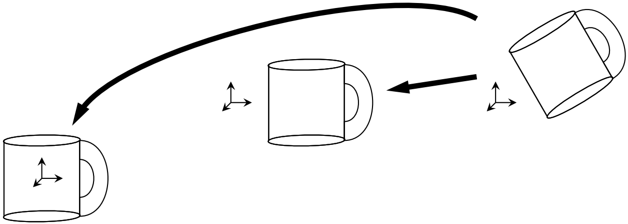
Inverting Composed Transforms

- Reverse order
  - $(R \circ T)^{-1}(\vec{p}')$



Inverting Composed Transforms

- Reverse order
  - $(R \circ T)^{-1}(\vec{p}') = \qquad R^{-1}(\vec{p}')$



Inverting Composed Transforms

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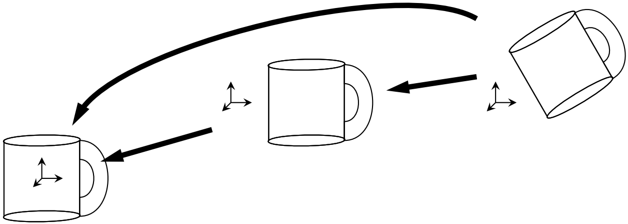
  - $(R \circ T)^{-1}(\vec{p}') = T^{-1}(R^{-1}(\vec{p}'))$



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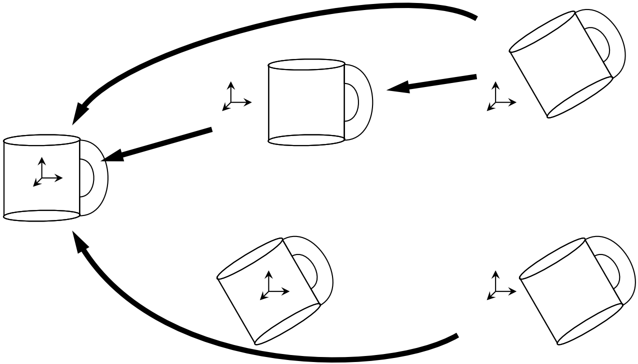

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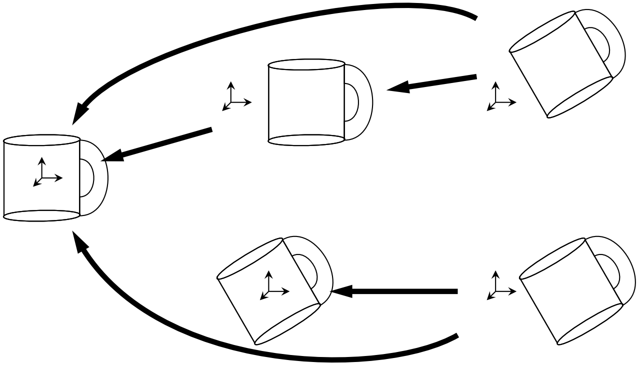

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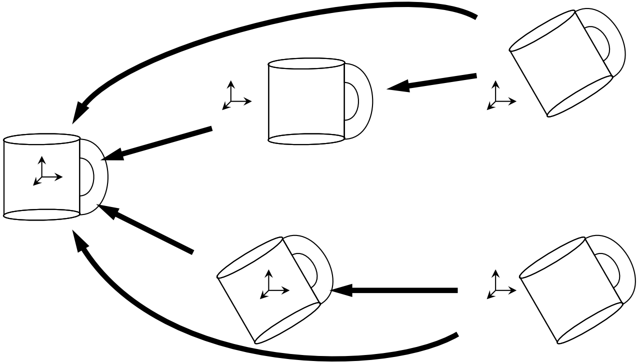

Translation

- $\vec{p}' = \vec{p} + \vec{t}$

Translation

- $\vec{p}' = \vec{p} + \vec{t}$
- $\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix} + \begin{bmatrix} t^x \\ t^y \\ t^z \end{bmatrix} = \begin{bmatrix} p^x + t^x \\ p^y + t^y \\ p^z + t^z \end{bmatrix}$

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- $\vec{t}$ says where $\vec{p}$-space origin ends up ($\vec{p}' = \vec{0} + \vec{t}$)

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- $\vec{t}$ says where $\vec{p}$-space origin ends up ($\vec{p}' = \vec{0} + \vec{t}$)
- Composition: $\vec{p}' = (\vec{p} + \vec{t_0}) + \vec{t_1} = \vec{p} + (\vec{t_0} + \vec{t_1})$

Linear Transforms

- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$

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- Matrix says where $\vec{p}$-space axes end up
  - $$\begin{bmatrix} a \\ d \\ g \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

Linear Transforms

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- Matrix says where $\vec{p}$-space axes end up
  - $$\begin{bmatrix} b \\ e \\ h \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$$

Linear Transforms

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- Matrix says where $\vec{p}$-space axes end up
  - $\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$

Linear Transforms

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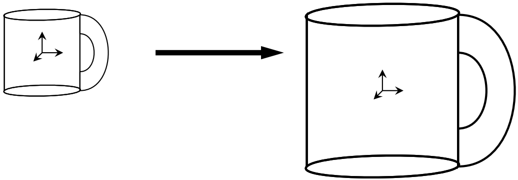
  - $$\begin{bmatrix} c \\ f \\ i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$
  - Composition: $\vec{p}' = M \, (N \, \vec{p}) = (M \, N)\vec{p}$

Common case: Scaling

- $\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} s_x p^x \\ s_y p^y \\ s_z p^z \end{bmatrix} = \begin{bmatrix} s_x & 0 & 0 \\ 0 & s_y & 0 \\ 0 & 0 & s_z \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$



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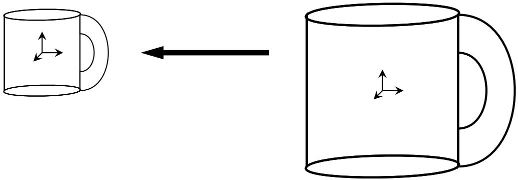
- Inverse: $\begin{bmatrix} 1/s_x & 0 & 0 \\ 0 & 1/s_y & 0 \\ 0 & 0 & 1/s_z \end{bmatrix}$



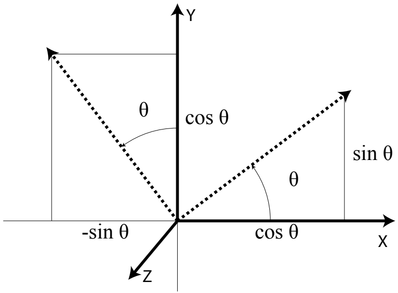
Common case: Reflection

- Negative scaling

Common case: Reflection

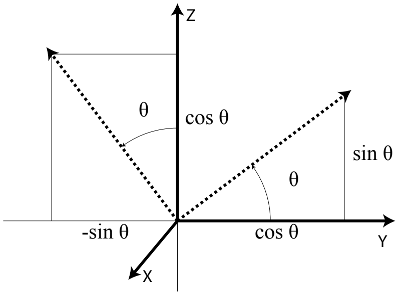
- Negative scaling
- $$\begin{bmatrix} p'^x \\ p'^y \\ p'^z \end{bmatrix} = \begin{bmatrix} -p^x \\ p^y \\ p^z \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} p^x \\ p^y \\ p^z \end{bmatrix}$$

Common case: Rotation



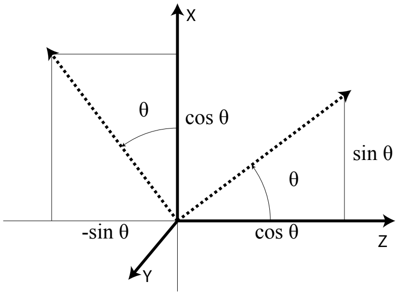
- Orthogonal, so $M^{-1} = M^T$
- Rotate around Z: $\vec{p}' = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \vec{p}$

Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$
- Rotate around X: $\vec{p}' = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & -\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \vec{p}$

Common case: Rotation



- Orthogonal, so $M^{-1} = M^T$
- Rotate around Y: $\vec{p}' = \begin{bmatrix} \cos\theta & 0 & \sin\theta \\ 0 & 1 & 0 \\ -\sin\theta & 0 & \cos\theta \end{bmatrix} \vec{p}$

Composing Transforms

- Scale by s along axis $\vec{a}$

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  - Rotate to align $\vec{a}$ with Z

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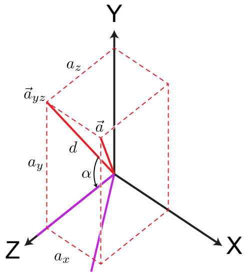
- Scale by s along axis $\vec{a}$
  - Rotate to align $\vec{a}$ with Z
  - Scale along Z

Composing Transforms

- Scale by s along axis $\vec{a}$
  - Rotate to align $\vec{a}$ with Z
  - Scale along Z
  - Rotate back

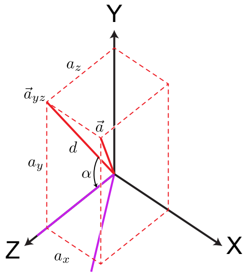
Rotate by α around X into XZ plane

- Projection of $\vec{a}$ onto YZ: $\vec{a}_{yz} = \begin{bmatrix} 0 \\ a^y \\ a^z \end{bmatrix}$



Rotate by α around X into XZ plane

- Projection of $\vec{a}$ onto YZ: $\vec{a}_{yz} = \begin{bmatrix} 0 \\ a^y \\ a^z \end{bmatrix}$
- length $d = \sqrt{(a^y)^2 + (a^z)^2}$



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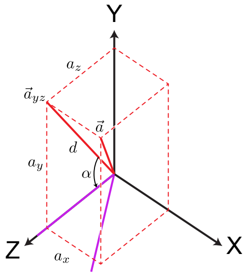
- So $\cos\alpha = a^z/d$, $\sin\alpha = a^y/d$



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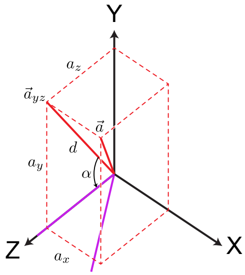
- So $\cos\alpha = a^z/d$, $\sin\alpha = a^y/d$
- $R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^z/d & -a^y/d \\ 0 & a^y/d & a^z/d \end{bmatrix}$



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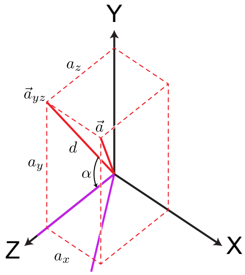
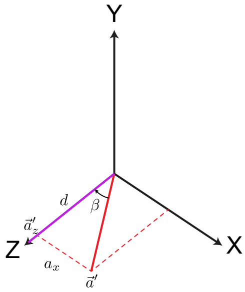
- So $\cos\alpha = a^z/d$, $\sin\alpha = a^y/d$
- $R_X = \begin{bmatrix} 1 & 0 & 0 \\ 0 & a^z/d & -a^y/d \\ 0 & a^y/d & a^z/d \end{bmatrix}$
- Result $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$



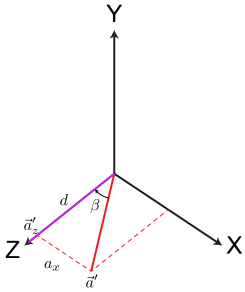
Rotate by -β around Y to Z axis

- $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$



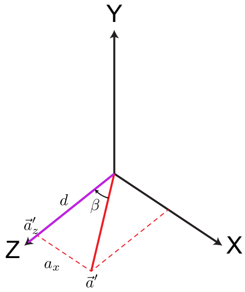
Rotate by -β around Y to Z axis

- $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$
- length $= 1$



Rotate by -β around Y to Z axis

- $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$
- length $= 1$
- So $\cos\beta = d$, $\sin\beta = a^x$



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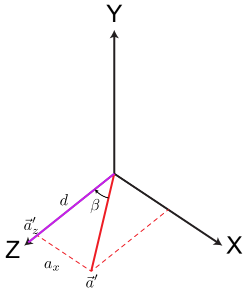
- $\vec{a}' = \begin{bmatrix} a^x \\ 0 \\ d \end{bmatrix}$
- length $= 1$
- So $\cos\beta = d$, $\sin\beta = a^x$
- $R_Y = \begin{bmatrix} d & 0 & -a^x \\ 0 & 1 & 0 \\ a^x & 0 & d \end{bmatrix}$



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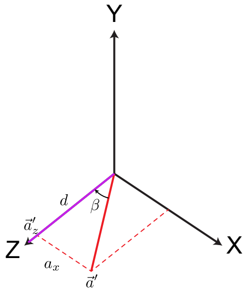
- length $= 1$
- So $\cos\beta = d$, $\sin\beta = a^x$
- $R_Y = \begin{bmatrix} d & 0 & -a^x \\ 0 & 1 & 0 \\ a^x & 0 & d \end{bmatrix}$
- Result $\vec{a}'' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$



Composing Transforms

- Scale by s along Z: $S_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$

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Composing Transforms

- Scale by s along Z: $S_Z = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & s \end{bmatrix}$
- Scale by s along axis $\vec{a}$
  - Rotate to align $\vec{a}$ with Z
  - Scale along Z
  - Rotate back
  - $\vec{p}' = R_X^{-1} R_Y^{-1} S_Z R_Y R_X \vec{p}$

Affine Transforms

- Affine = Linear + Translation

Affine Transforms

- Affine = Linear + Translation
- Composition? $A\ (B\ \vec{p} + \vec{t_0}) + \vec{t_1} = A\ B\ \vec{p} + A\ \vec{t_0} + \vec{t_1}$
- Yuck!

Homogeneous Coordinates

- Add a '1' to each point

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- $1 = (0p^x + 0p^y + 0p^z) + 1$

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  - $\vec{x}$, $\vec{y}$, $\vec{z}$ say where the $\vec{p}$-space axes end up
- Composition: Just matrix multiplies!

Composing Transforms

- Rotate by θ about line between $\vec{p}_0$ and $\vec{p}_1$:

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- T

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    - Rotate by θ around Z
- $$R_Z(\theta) R_Y R_X T$$

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  - Rotate to align $\vec{p}_1 - \vec{p}_0$ with Z
  - Rotate by θ around Z
  - Undo $\vec{p}_1 - \vec{p}_0$ rotation
- $$R_X^{-1} R_Y^{-1} R_Z(\theta) R_Y R_X T$$

Composing Transforms

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    - Translate $\vec{p}_0$ to origin
    - Rotate to align $\vec{p}_1 - \vec{p}_0$ with Z
    - Rotate by θ around Z
    - Undo $\vec{p}_1 - \vec{p}_0$ rotation
    - Undo translation
- $T^{-1} R_X^{-1} R_Y^{-1} R_Z(\theta) R_Y R_X T$

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  - $J = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}$
- *Upper-left 3x3*

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- OR multiply *column* normal by inverse transpose

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- $\vec{n'} = \vec{n} J^{-1}$
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- OR multiply *column* normal by inverse transpose
  - $(J^{-1})^T = J$ if J is orthogonal (only rotations)