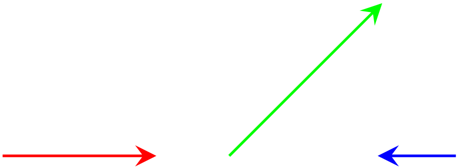
Vector Math

CMSC 435/634

Abstract Vectors

($\vec{u}$, $\vec{v}$, $\vec{w}$ vectors; a, b, c scalars)

- $\vec{u} + \vec{v}$ is a vector
- $a\vec{u}$ is a vector
- $\vec{u} + \vec{v} = \vec{v} + \vec{u}$
- $(a + b)\vec{u} = a\vec{u} + b\vec{u}$
- $(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$



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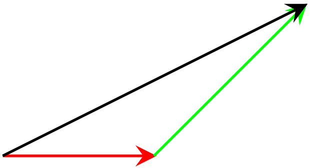

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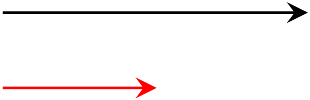

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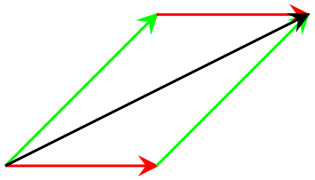

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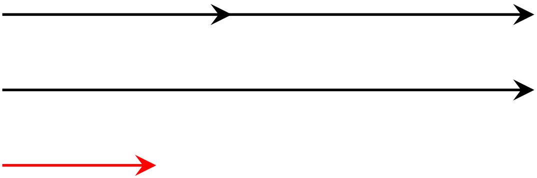

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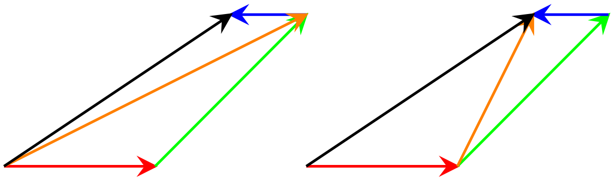
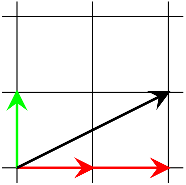


Basis Vectors

Vector as linear combination of *basis vectors*

- $\vec{v} = 2\hat{i} + 1\hat{j} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
- $\vec{v} = 1\hat{m} + 2\hat{n} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
- $\vec{v} = 1\hat{p} + 1\hat{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$



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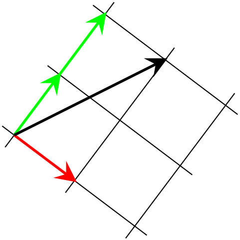

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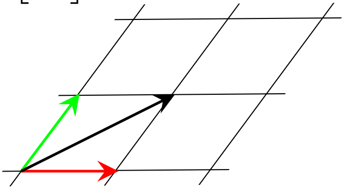

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- $\vec{v} = 1\hat{p} + 1\hat{q} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$
- Column: $\vec{v} = \begin{bmatrix} v^0 \\ v^1 \end{bmatrix}$ (we'll usually use this form)
- Row: $\vec{v} = \begin{bmatrix} v_0 & v_1 \end{bmatrix}$ (some texts; I like for normals)

Matrices

- Matrix: $A = \begin{bmatrix} a_0^0 & a_1^0 \\ a_0^1 & a_1^1 \end{bmatrix} = \left[a_j^i \right]$
- Transpose: $A^T = \begin{bmatrix} a_0^0 & a_0^1 \\ a_1^0 & a_1^1 \end{bmatrix} = \left[a_i^j \right]$
- Multiply: $AB = \begin{bmatrix} a_0^0 & a_1^0 \\ a_0^1 & a_1^1 \end{bmatrix} \begin{bmatrix} b_0^0 & b_1^0 \\ b_0^1 & b_1^1 \end{bmatrix} =$
$\begin{bmatrix} a_0^0 b_0^0 + a_1^0 b_0^1 & a_0^0 b_1^0 + a_1^0 b_1^1 \\ a_0^1 b_0^0 + a_1^1 b_0^1 & a_0^1 b_1^0 + a_1^1 b_1^1 \end{bmatrix} = \left[a_\alpha^i b_j^\alpha \right]$

Adjoint and Inverse

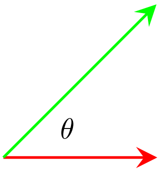
- Inverse: $A^{-1}A = AA^{-1} = I$
- Determinant: $|A|$
  - $|a| = a$
  - $\begin{vmatrix} a & b \\ c & d \end{vmatrix} = a|d| - b|c|$
  - $\begin{vmatrix} a & b & c \\ c & d & e \\ f & g & h \end{vmatrix} = a \begin{vmatrix} d & e \\ g & h \end{vmatrix} - b \begin{vmatrix} c & e \\ f & h \end{vmatrix} + c \begin{vmatrix} c & d \\ f & g \end{vmatrix}$
- Adjoint: $A^* = cof(A)^T$ (matrix of cofactors $cof(A)$)
- $A^{-1} = \frac{A^*}{|A|}$

Dot Product

- Also called inner product
  - $\vec{u} \bullet \vec{v}$ is a scalar
  - $\vec{u} \bullet \vec{v} = \vec{v} \bullet \vec{u}$
  - $(a\vec{u}) \bullet \vec{v} = a(\vec{u} \bullet \vec{v})$
  - $(\vec{u} + \vec{v}) \bullet \vec{w} = \vec{u} \bullet \vec{w} + \vec{v} \bullet \vec{w}$
  - $\vec{v} \bullet \vec{v} \geq 0$
  - $\vec{v} \bullet \vec{v} = 0 \leftrightarrow \vec{v} = \vec{0}$
- Matrix notation: $\vec{u} \bullet \vec{v} = U^T V = u_\alpha v^\alpha$

Dot Product as Norm

- $\vec{v} \bullet \vec{v} = |\vec{v}|^2$
- $\vec{u} \bullet \vec{v} = |\vec{u}||\vec{v}| \cos\theta$
  - Defines angle θ!
  - If $|\vec{v}| = 1$, gives projection of $\vec{u}$ onto $\vec{v}$
  - If $|\vec{u}| = |\vec{v}| = 1$, gives just $\cos\theta$



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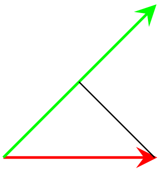

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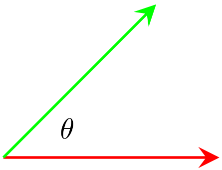


Orthogonal & Normal

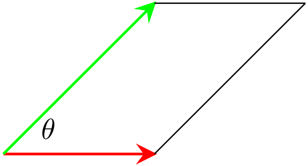
- *Orthogonal* = perpendicular: $\vec{u} \bullet \vec{v} = 0$
- *Normal* (this usage) = unit-length: $\vec{u} \bullet \vec{u} = 1$
- *Orthonormal*: set of vectors both orthogonal and normal
- *Orthogonal matrix*: rows (& columns) **orthonormal**
  - For orthogonal matrices, $A^{-1} = A^T$

3D Cross Product

$\vec{u} \times \vec{v}$

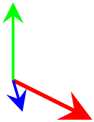
- length = area of parallelogram = twice area of triangle
  - $|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}| \sin(\theta)$
- direction = perpendicular to $\vec{u}$ and $\vec{v}$ (right hand rule)

$$\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & & \\ \hat{j} & U & V \\ \hat{k} & & \end{vmatrix} = \begin{bmatrix} u^1 v^2 - u^2 v^1 \\ u^2 v^0 - u^0 v^2 \\ u^0 v^1 - u^1 v^0 \end{bmatrix}$$



Building an Orthogonal Basis

Vectors $\vec{u}$, $\vec{v}$, $\vec{w}$



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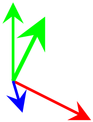

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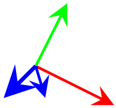

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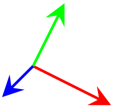

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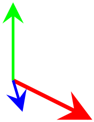

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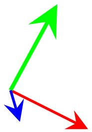

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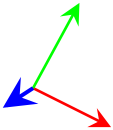

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