Set Theory

- Set Theory
- Properties of Sets
- Disproofs and Algebraic Proofs

Reading (Epp's textbook)
6.1 – 6.3

Set Theory

- A set is a collection of elements.
- If S is a set,
  - The notation $x \in S$ means that x is an element of S.
  - The notation $x \notin S$ means that x is not an element of S.
- There is only one set with no elements, named the **empty set** and denoted by the symbol $\emptyset$.
- If *A* and *B* are sets, then *A* is called as subset of *B*, written $A \subseteq B$, if, and only if, every element of *A* is also an element of *B*.
- $\emptyset \subseteq$ A for any set A.
- Let *A* and *B* be sets. *A* is a proper subset of *B,* if, and only if, ($\Leftrightarrow$) 1) every element of *A* is in *B* ($A \subseteq B$), 2) but there is at least one element of *B* that is not in *A*.
- If A $\subseteq$ B, then B is called a superset of A, written B $\supseteq$ A

Subsets: Proving Set Equality

- Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.

  Symbolically: $A = B \;\Leftrightarrow\; A \subseteq B$ and $B \subseteq A$

- Example:

$$A = \{m \in Z \mid m = 2a \; for\ some\ a \in Z\}$$
$$B = \{n \in Z \mid n = 2b - 2 \; for\ some\ b \in Z\}$$

- Prove that $A \subseteq B$. (Method of Generalization)
  1. Suppose x is a particular but arbitrarily chosen element of A.
  2. We must show that $x \in B$. By definition of B, this means we must show that $x = 2 \times (some\ integer) - 2$.
     i. $x = 2a$.
     ii. Let $b = a + 1$. Does $b \in Z$?
     iii. Now we must check that $x = 2 \times b - 2$.

Subsets: Proving Set Equality

- Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.

  Symbolically: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$

- Example:

$$A = \{m \in Z \mid m = 2a \text{ } for \text{ } some \text{ } a \in Z\}$$
$$B = \{n \in Z \mid n = 2b - 2 \text{ } for \text{ } some \text{ } b \in Z\}$$

- Prove that B $\subseteq A$. (Method of Generalization)
  1. Suppose x is a particular but arbitrarily chosen element of B.
  2. We must show that $x \in A$. By definition of A, this means we must show that $x = 2 \times (some\ integer)$.
     - *i.* $x = 2b - 2$.
     - ii. Let a $= b - 1$. Does a $\in$ Z?
     - iii. Now we must check that $x = 2 \times a$.

Partitions of Sets

- Two sets are called **disjoint** if, and only if, they have no elements in common.

  Symbolically: A and B are disjoint ⇔ $A \cap B = \emptyset$.

- Sets A_1, A_2, A_3 … are **mutually disjoint** if, and only if, no two sets A_i and A_j with distinct subscripts have any element in common. More precisely, for all *i, j* = 1, 2, 3, …

  $$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

- A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \ldots\}$ is a **partition** of set A if, and only if,

  1. A is the union of all the A_i
  2. The sets A_1, A_2, A_3 … are mutually disjoint.

Power Sets

- P(A) "power set of A"
- P(A) = {B | B $\subseteq$ A} (contains all subsets of A)
- Examples:
  - A = {x, y, z}

    P(A) = {$\varnothing$, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}}
  - A = $\varnothing$

    P(A) = {$\varnothing$}
- If a set S contains n distinct elements, $n \in \boldsymbol{N}$, we call S a finite set with cardinality $|S| = n$.

Cardinality of Power Sets

- Cardinality of power sets:
  - $|P(A)| = 2^{|A|}$

- Imagine each element in A has an "on/off" switch
- Each possible switch configuration in A corresponds to one element in 2^A

A	1	2	3	4	5	6	7	8
x	x	x	x	x	x	x	x	x
y	y	y	y	y	y	y	y	y
z	z	z	z	z	z	z	z	z

Cartesian Products

- The symbol (a, b) denotes the ordered pair (ordered two-tuple). $(a, b) = (c, d)$ means that $a = c$ and $b = d$.
- In general two ordered n-tuples $(x_1, x_2, \dots, x_n)$ and $(y_1, y_2, \dots, y_n)$ are equal if, and only if, $x_1 = y_1, x_2 = y_2, \dots, x_n = y_n$.

Note that:

- $A \times \emptyset = \emptyset$
- $\emptyset \times A = \emptyset$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- If $|A| = n$ and $|B| = m$, $|A \times B|$ is $n \times m$

- **Cartesian product of two or more sets is defined as**:
  $A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \dots a_n) \mid a_1 \in A_1, a_2 \in A_2, \dots, a_n \in A_n\}.$

Operations on Sets

Let sets A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.

   Symbolically: $A \cup B = \{x \in U \mid x \in A \text{ } or \text{ } x \in B\}$.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.

   Symbolically: $A \cap B = \{x \in U \mid x \in A \text{ } and \text{ } x \in B\}$.

3. The **difference** of B minus A, denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.

   Symbolically: $B - A = \{x \in U \mid x \in B \text{ } and \text{ } x \notin A\}$.

4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

   Symbolically: $A^c = \{x \in U \mid x \notin A\}$.

Properties of Sets

Theorem 6.2.1 Some Subset Relations

- *Inclusion of Intersection:* For all sets A and B,

  (a) $A \cap B \subseteq A$ and (b) $A \cap B \subseteq B$.
- *Inclusion in Union:* For all sets A and B,

  (a) $A \subseteq A \cup B$ and (b) $B \subseteq A \cup B$.
- *Transitive Property of Subsets:* For all sets A, B, and C,

  if $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Properties of Sets

An **identity** is an equation that is universally true for all elements in some set.

- **Theorem 6.2.2 Set Identities (page 355)**

Theorem 6.2.2(3)- *Distributive Laws:* For all sets, *A, B*, and *C,*

(a) $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$ and

(b) $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.

Theorem 6.2.2(9) - *De Morgan's Laws:* For all sets *A* and *B*,

(a) $(A \cup B)^c = A^c \cap B^c$ and

(b) $(A \cap B)^c = A^c \cup B^c$.

Proving Distributivity of ∪

Prove that for all sets, *A, B*, and *C*,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof:

Suppose *A, B, and C* are sets.

1. Proof that $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$:

Case 1 ($x \in A$): Since $x \in A$, then $x \in A \cup B$ by definition of union **and** also $x \in A \cup C$ by definition of union.

Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of intersection.

Case 2 ($x \in B \cap C$): Since $x \in B \cap C$, then $x \in B$ **and** $x \in C$ by definition of intersection.

Since $x \in B$, $x \in A \cup B$ **and** since $x \in C$, $x \in A \cup C$, both by definition of union.

Hence $x \in (A \cup B) \cap (A \cup C)$ by definition of intersection.

In both cases, $x \in (A \cup B) \cap (A \cup C)$.

Hence $A \cup (B \cap C) \subseteq (A \cup B) \cap (A \cup C)$ by **definition of subset.**

Proving Distributivity of ∪ (Cont.)

Prove that for all sets, A, B, and C,

$$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$$

Proof continues:

2. Proof that $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$:

Suppose $x \in (A \cup B) \cap (A \cup C)$. By definition of intersection, $x \in A \cup B$ and $x \in A \cup C$. Consider the two cases $x \in A$ and $x \notin A$.

Case 1 ($x \in A$): Since $x \in A$, we can immediately conclude that $x \in A \cup (B \cap C)$ by definition of union.

Case 2 ($x \notin A$): Since $x \in A \cup B$, x is in at least one of A or B. But x is not in A; hence x is in B. Similarly, since $x \in A \cup C$, x is in at least one of A or C. But x is not in A; hence x is in C. We have shown that both $x \in B$ and $x \in C$, and so by definition of intersection, $x \in B \cap C$. It follows by definition of union that $x \in A \cup (B \cap C)$.

In both cases $x \in A \cup (B \cap C)$. Hence, by definition of subset, $(A \cup B) \cap (A \cup C) \subseteq A \cup (B \cap C)$.

Conclusion: From 1 and 2 we proved that $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.

De Morgan's Law

Prove that for all sets, *A, B*, and *C*,

$$(A \cap B)^c = A^c \cup B^c$$

Proof:

Suppose *A* and *B* are sets.

1. Proof that $(\boldsymbol{A} \cap \boldsymbol{B})^{\boldsymbol{c}} \subseteq \boldsymbol{A}^{\boldsymbol{c}} \cup \boldsymbol{B}^{\boldsymbol{c}}$

If $x \in (A \cap B)^c$. *[We must show that* $x \in A^c \cup B^c$*.]*

By definition of complement, $x \notin A \cap B$.

But to say that $x \notin A \cap B$ means that

it is false that (*x* is in *A* and *x* is in *B*).

By De Morgan's laws of **logic**, this implies that

x is not in *A* or *x* is not in *B*,

which can be written $x \notin A$ or $x \notin B$.

Hence $x \in A^c$ or $x \in B^c$ by definition of complement. It follows, by definition of union, that $x \in A^c \cup B^c$ *[as was to be shown].*

So $(A \cap B)^c \subseteq A^c \cup B^c$ by definition of subset.

De Morgan's Law

Proof continues:

2. Proof that $\boldsymbol{A^c \cup B^c \subseteq (A \cap B)^c}$

If $x \in A^c \cup B^c$. *[We must show that $x \in (A \cap B)^c$.]*

By definition of union, $x \in A^c$ or $x \in B^c$

By definition of complement, $x \notin A$ or $x \notin B$.

In other words

it is false that (x is in A or x is in B).

By De Morgan's laws of **logic**, this implies that

x is not in A and x is not in B,

By definition of intersection can be written $x \notin A \cap B$.

Hence $x \in (A \cap B)^c$ by definition of complement *[as was to be shown].*

So $A^c \cup B^c \subseteq (A \cap B)^c$ by definition of subset.

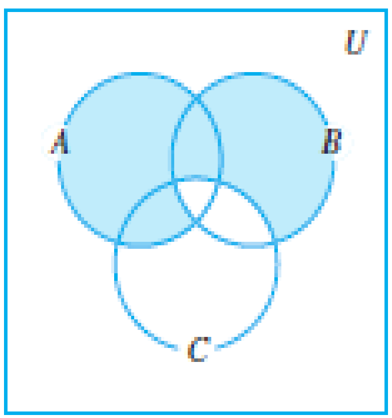
Conclusion: Since both set containments, (1) and (2), have been proved, $(A \cap B)^c = A^c \cup B^c$ by definition of set equality.

Disproving an Alleged Set Property

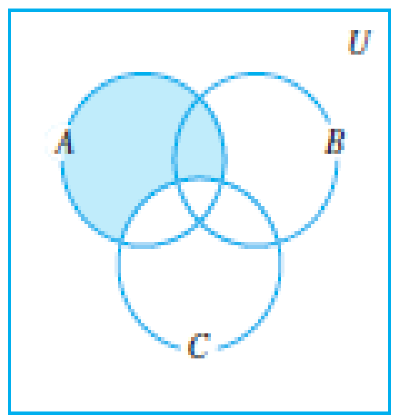
Is the following set property true?

For all sets A, B, and C, $(A - B) \cup (B - C) = A - C$.

Disprove: Recall that to show a universal statement is false, it suffices to find a counterexample for which it is false.



$(A - B) \cup (B - C)$



$A - C$

Construct a concrete counterexample in order to confirm your answer.

Algebraic Proofs of Set Identities

Construct an algebraic proof that for all sets A, B, and C,

$$(A \cup B) - C = (A - C) \cup (B - C).$$

Solution:

Let A, B, and C be any sets. Then:

$(A \cup B) - C = (A \cup B) \cap C^c$ by the set difference law

$= C^c \cap (A \cup B)$ by the commutative law for $\cap$

$= (C^c \cap A) \cup (C^c \cap B)$ by the distributive law

$= (A \cap C^c) \cup (B \cap C^c)$ by the commutative law for $\cap$

$= (A - C) \cup (B - C)$ by the set difference law.