Lecture 3

- Logic
  - Predicates and Quantified Statements
  - Statements with Multiple Quantifiers

Reading (Epp's textbook)
3.1 - 3.3

- Introduction to Proofs

Reading (Epp's textbook)
4.1-4.2

Propositional Functions

- A **predicate** is a sentence that contains a finite number of variables and becomes a statement when specific values are substituted for the variables.
  - **e.g.: x -3 > 5.**
- Let us call this propositional function P(x), where P is the predicate (x-3 >5) and x is the variable.

What is the truth value of P(2) ? false

What is the truth value of P(8) ? false

What is the truth value of P(9) ? true

Propositional Functions

- Let us consider the propositional function $Q(x, y, z)$ defined as:
  - $x + y = z$.

- Here, Q is the **predicate** and x, y, and z are the variables.

What is the truth value of $Q(2, 3, 5)$? true

What is the truth value of $Q(0, 1, 2)$? false

What is the truth value of $Q(9, -9, 0)$? true

Truth Set

- If P(x) is a propositional function and *x* has domain *D*

- The **truth set** of P(x) is denoted
  - $\{x \ \in D \mid P(x)\}$
  - Note: $\forall x \in in\ the\ truth\ set\ of\ P(x), P(x)$ should be true.

Universal Quantification

- Let P(x) be a propositional function.

- **Universally quantified sentence**:
  - For all x in the universe of discourse P(x) is true.

- Using the universal quantifier ∀:
  - $\forall x \in D, P(x)$
  - Note: $\forall x \in D, P(x)$ is either true or false, so it is a statement (proposition), not a propositional function.

- A value for x for which $P(x)$ is false is called a **counterexample** to the universal statement.

  - Statement: "All birds can fly."
  - Disproved by counterexample: Penguin.

Universal Quantification

- Example:
  - S(x): x is a UMBC student.
  - G(x): x is a genius.

- What does $\forall x\,, (S(x) \rightarrow G(x))$ mean ?

"If x is a UMBC student, then x is a genius."

or

"All UMBC students are geniuses."

Existential Quantification

- **Existentially quantified sentence**:
  - There exists an x in the universe of discourse for which P(x) is true.

- Using the existential quantifier ∃:
  - $\exists x \in D \text{ such that } P(x)$

  "There is an x such that P(x)."

  "There is at least one x such that P(x)."

- Note: $\exists x \in D \text{ such that } P(x)$ is either true or false, so it is a statement (proposition), but no propositional function.

Existential Quantification

- Example:
  - P(x): x is a UMBC professor.
  - G(x): x is a genius.

- What does $\exists x$ such that $(P(x) \land G(x))$ mean ?

"There is an x such that x is a UMBC professor and x is a genius."

or

"At least one UMBC professor is a genius."

Negation

- Negation of a Universal Statement

  $\neg\ (\forall x\ \in D, P(x))$ is logically equivalent to

  $\exists x\ \in D\ such\ that\ \neg P(x)$.

- Negation of an Existential Statement

  $\neg(\exists x\ \in D\ such\ that\ P(x))$ is logically equivalent to

  $\forall x \in D, \neg P(x)$.

Negation of a Universal Cond. Statement

$\neg\ (\forall x, if\ P(x)\ then\ Q(x))$ is logically equivalent to
$\exists x\ such\ that\ P(x)\ and\ \neg Q(x)$.

- Proof

$\neg\ \left(\forall x, if\ P(x)\ then\ Q(x)\right) \equiv\ \exists x\ such\ that, \neg(P(x) \rightarrow Q(x))$.

But

$\sim\left(P(x) \rightarrow Q(x)\right) \equiv$ P(x) $\wedge \sim Q(x)$

Negation of a Multiply-Quantified Statements

$\sim(\forall x \; in \; D, \exists y \; in \; E \; such \; that \; P(x,y))$ is logically equivalent to

$$\exists x \; in \; D \; such \; that \; \forall y \; in \; E, \sim P(x,y).$$

And

$\sim(\exists x \; in \; D \; such \; that \;\; \forall y \; in \; E, \;\; P(x,y))$ is logically equivalent to

$$\forall x \; in \; D, \exists y \; in \; E \; such \; that \; \sim P(x,y).$$

Methods of Proof: Some Definitions

- For all objects A, B, and C, (1) $A = A$, (2) if $A = B$ then $B = A$, and (3) if $A = B$ and $B = C$, then $A = C$.
- The set of all integers is closed under addition, subtraction, and multiplication.
- If n is an integer, then

  n is even $\Leftrightarrow \exists$ an integer k such that $n = 2k$.

  n is odd $\Leftrightarrow \exists$ an integer k such that $n = 2k + 1$.
- n is prime $\Leftrightarrow \forall$ positive integers r and s, if $n = rs$ then either $r = 1$ and $s = n$ or $r = n$ and $s = 1$.
- n is composite $\Leftrightarrow \exists$ positive integers r and s such that $n = rs$ and $1 < r < n$ and $1 < s < n$.

Proving Existential Statements

- A statement in the form:

$$\exists x \in D \; such \; that \; P(x)$$

is true if, and only if,

$P(x)$ is true for at least one x in *D.*

- Example
  - Prove the following: ∃ an even integer *n* that can be written in two ways as a sum of two prime numbers.
  - Solution: Let *n* = 10. Then 10 = 5 + 5 = 3 + 7 and 3, 5, and 7 are all prime numbers.

- This method is called **constructive proof of existence**!

Proving Universal Statements

To prove a statement of the form

$\forall x$ in D, if $P(x)$ then $Q(x)$

I. When D is finite or when only a finite number of elements satisfy $P(x)$, such a statement can be proved by the method of exhaustion.

II. Method of Generalizing from the Generic Particular: suppose x is a *particular* but *arbitrarily chosen* element of the set, and show that x satisfies the property.

When the method of generalizing from the generic particular is applied to a property of the form "If $P(x)$ then $Q(x)$," the result is the ***method of direct proof.***

Disproving a Statement

- To disprove a universal statement of the form:

  $\forall x$ in D, if $P(x)$ then $Q(x)$.

  we should show that the negation of the statement is true

  $\exists x$ in D such that $P(x)$ and not $Q(x)$.

- Such an x is called a **counterexample**.
- To disprove an existential statement of the form:

  $\exists x$ in D such that $P(x)$.

  we should show that the negation of the statement is true

  $\forall x$ in $D, \neg P(x)$.

- Use the **Method of Generalizing from the Generic Particular.**

Disproving a Statement: Example

- Disprove the universal statement:

  $\forall a, b \in R$, if $a^2 = b^2$ then $a = b$.

Solution: We should show that the negation of the statement is true.

$\exists a, b \in R$ such that $a^2 = b^2$ and not $a = b$.

Counterexample: Let $a = 1$ and $b = -1$.

Disproving a Statement: Example

- Disprove the existential statement:

  $\exists x$ in Z^+ such that $x^2 + 3x + 2$ is prime.

Solution: We should show that the negation of the statement is true

$\forall x \in Z^+, x^2 + 3x + 2$ is not prime (composite).

Method of Generalizing from the Generic Particular:

Suppose x is any (particular but arbitrarily chosen) positive integer. Then $x^2 + 3x + 2 = (x+1)(x+2)$. Both $(x+1) and$ $(x+2)$ are integers and are > 1 (since $x \geq 1$). Thus $x^2 + 3x + 2$ is a product of two integers each greater than 1, and so $x^2 + 3x + 2$ is not prime.

Method of Direct Proof

A statement in the form $\forall x$ in D, p $\rightarrow$ q can be proved by showing that if p is true, then q is also true.

Example: Give a direct proof of the theorem "$\forall n$ in Z, if n is odd, then n^2 is odd."

Idea: Assume that the hypothesis of this implication is true (n is odd). Then use rules of inference and known theorems to show that q must also be true (n^2 is odd).

Direct Proof Example

- n is odd.

- Then $n = 2k + 1$, where k is an integer.

- Consequently, $n^2 = (2k + 1)^2$.

$$
\begin{aligned}
&= 4k^2 + 4k + 1 \\
&= 2(2k^2 + 2k) + 1
\end{aligned}
$$

✓ Since n^2 can be written in this form, it is odd.

Indirect Proof

A statement in the form $\forall x$ in D, p $\rightarrow$ q is equivalent to its **contra-positive** $\neg$q $\rightarrow$ $\neg$p.

- Therefore, we can prove p $\rightarrow$ q by showing that whenever q is false, then p is also false.

Example:

Give an indirect proof of the theorem "$\forall n$ in Z, if 3n + 2 is odd, then n is odd."

Idea: Assume that the conclusion of this statement is false (n is even). Then use rules of inference and known theorems to show that p must also be false (3n + 2 is even).

Indirect Proof Example

- n is even.
- Then $n = 2k$, where k is an integer.
- It follows that
$$3n + 2 = 3(2k) + 2 = 6k + 2 = 2(3k + 1)$$
- Therefore, $3n + 2$ is even.

✓ We have shown that the contrapositive of the statement is true, so the implication itself is also true (If $2n + 3$ is odd, then n is odd).

Rational Numbers

- A real number *r* is rational ⇔ ∃ integers $a\ and\ b$ such that $r = \frac{a}{b}\ and\ b \neq 0$.
- Every integer is a rational number.
- The sum of any two rational numbers is rational.
- The double of a rational is rational.