Graphs

- Introduction to Graphs
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- Connected Graphs
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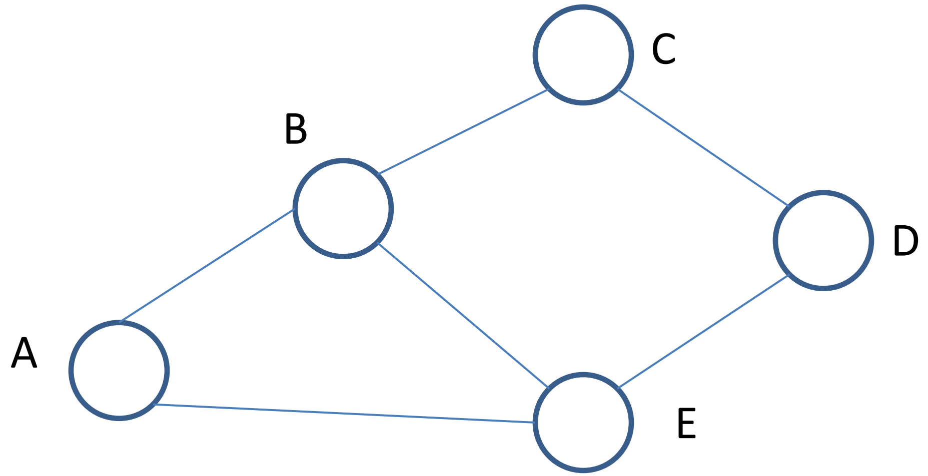
Reading (Epp's textbook)
10.1- 10.3

Introduction to Graphs

- **Graph** is a fundamental mathematical structure in computer science
- A **graph** $G = (V, E)$ consists of V, a nonempty set of vertices, and E, a set of edges.
- Lots of applications in many areas: web search, networking, databases, . . .

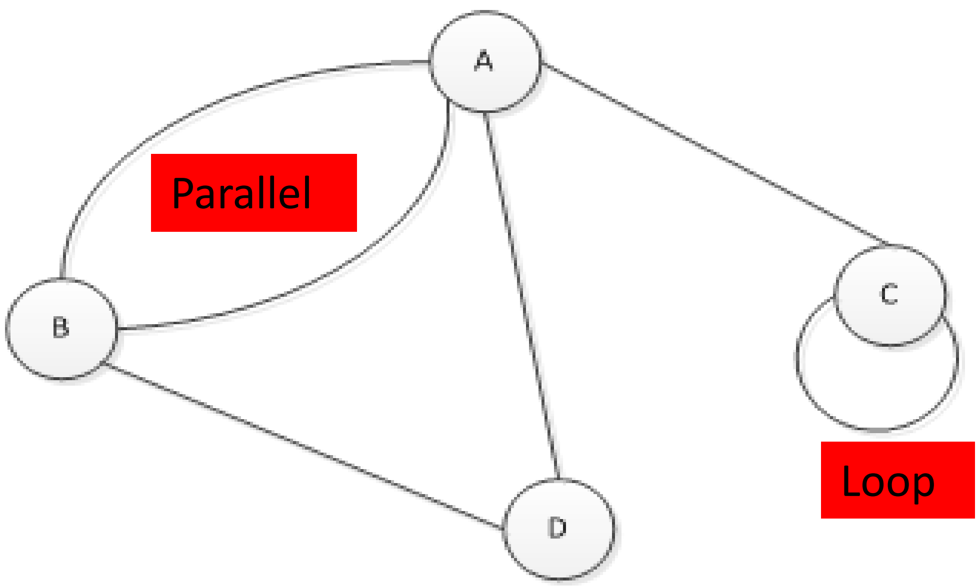


$V = \{ A, B, C, D, E \}$

$E = \{ \{A, B\}, \{A,E\}, \{B,C\}, \{B,E\}, \{C,D\}, \{E,D\} \}$

Introduction to Graphs

- For each edge $e \in E$, $e = \{u, v\}$ where $u, v \in V$. The vertices u, v called **endpoints** of edge e.
- An edge $e \in E$, $e = \{v, v\}$ for some $v \in V$, is called a **loop**.
- Two or more distinct edges with the same set of endpoints are said to be **parallel**.

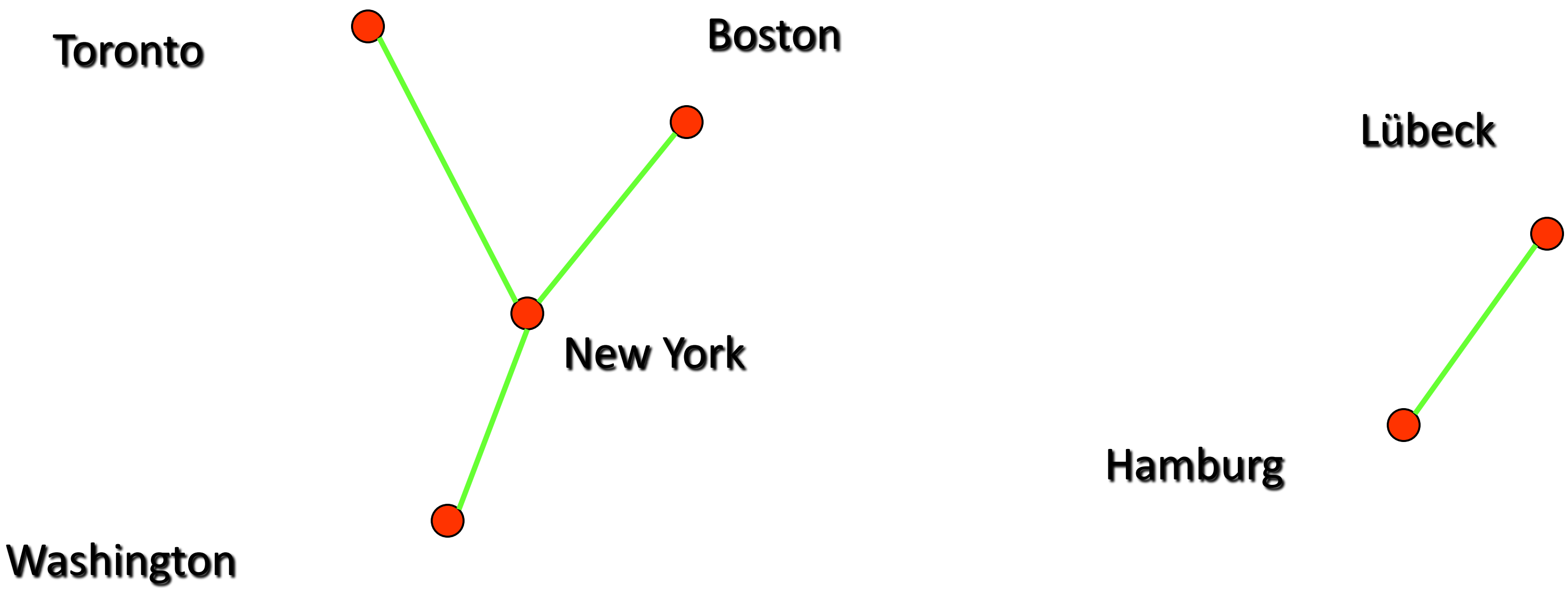


Graph Models Example

- A **simple graph** is a graph that **does not** have any loops or parallel edges, and no specified direction on its edges.

How can we represent a network of (bi-directional) railways connecting a set of cities?

- We should use a **simple graph** with an edge {a, b} indicating a train connection between cities a and b.

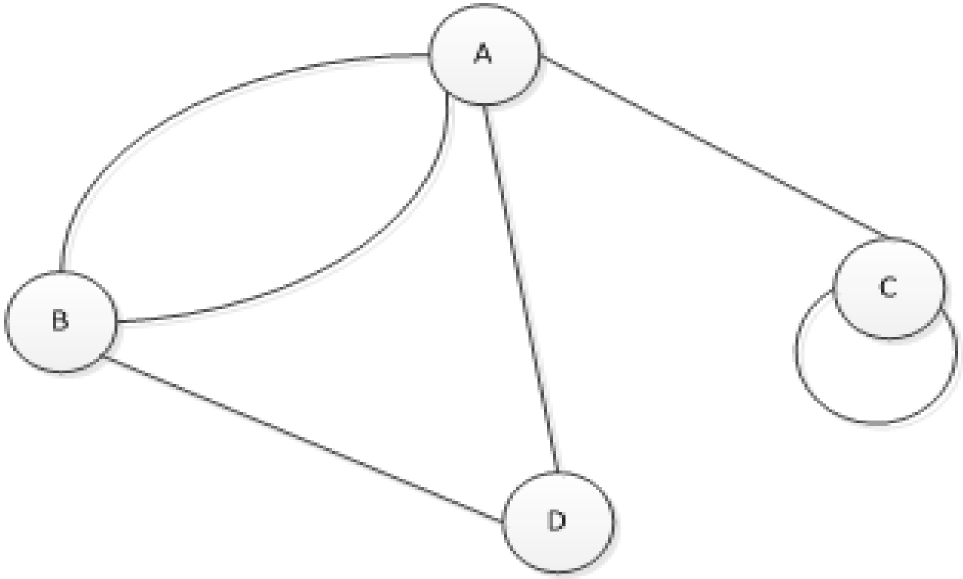


Graph Terminology

- Two vertices u and v in an undirected graph G are called **adjacent** (or **neighbors**) in G if {u, v} is an edge in G.
- **Neighborhood** of a vertex is the set of vertices adjacent to it.
- If e = {u, v}, the edge e is called **incident with** the vertices u and v. The edge e is also said to **connect** u and v.
- A vertex with no neighbors is called **isolated**, since it is not adjacent to any vertex.
- **Note:** A vertex with a **loop** is **not isolated**, even if it is not adjacent to any **other** vertex.

Graph Terminology

- The **degree** of a vertex in an undirected graph is the number of edges incident with it, except that a loop at a vertex contributes twice to the degree of that vertex.
- The degree of the vertex v is denoted by **deg(v).**
- The **total degree of G** is the sum of the degrees of all the vertices in G.



deg(A) = 4
deg(C) = 3
Total degree of G = 12

Question

Consider a simple graph G where two vertices A and B have the **same neighborhood**. Which of the following statements must be true about G?

A. The degree of each vertex must be even.

B. Both A and B have a degree of 0.

C. There cannot be an edge between A and B.

The Handshaking Theorem

- Let G = (V ,E) be an undirected graph with *e* edges. Then

$$\sum_{v \in V} \deg(v) = 2e$$

Intuition: Each edge contributes two to the sum of the degrees!

Example: How many edges are there in a graph with 10 vertices, each of degree 6?

Solution: The sum of the degrees of the vertices is 6·10 = 60. According to the Handshaking Theorem, it follows that 2e = 60, so there are 30 edges.

Questions in class

- Is it possible to construct a graph with 5 vertices where each vertex has degree 3?

- Is it possible to construct a simple graph with four vertices of degrees 1, 1, 3, and 3?

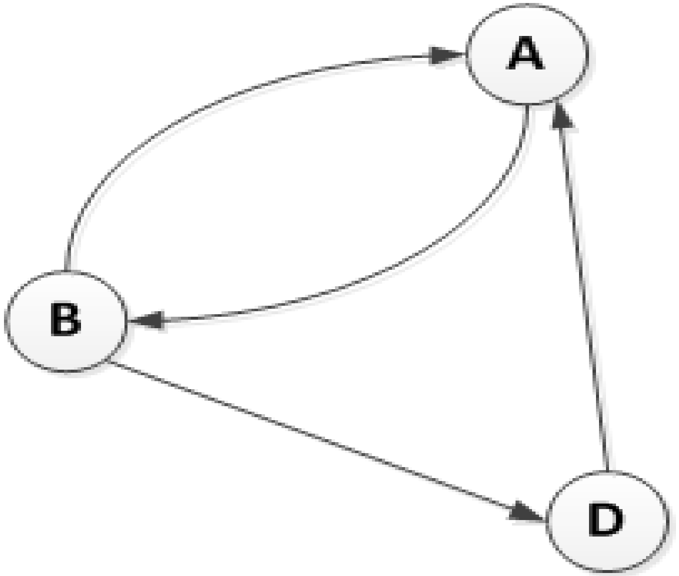
Directed Graphs

Definitions:

- A **directed graph** G = (V, E) consists of a set V of vertices and a set E of edges that are **ordered pairs** of elements in V.
  - For each $e \in E$, $e = (u, v)$ where $u, v \in V$.

V = { A, B, D }

E = { (A, B), (B, A), (B, D), (D, A) }



Directed Graph Terminology

- When (u, v) is an edge of the directed graph, u is said to be **adjacent to** v, and v is said to be **adjacent from** u.
- The vertex u is called the **initial vertex** of (u, v), and v is called the **terminal vertex** of (u, v).
- The initial vertex and terminal vertex of a loop are the same!
- In a directed graph, the **in-degree** of a vertex v, denoted by $\boldsymbol{deg^{-}(v)}$, is the number of edges with v as their **terminal vertex**.
- The **out-degree** of v, denoted by $deg^{+}(v)$, is the number of edges with v as their **initial vertex**.

Directed Graph Example

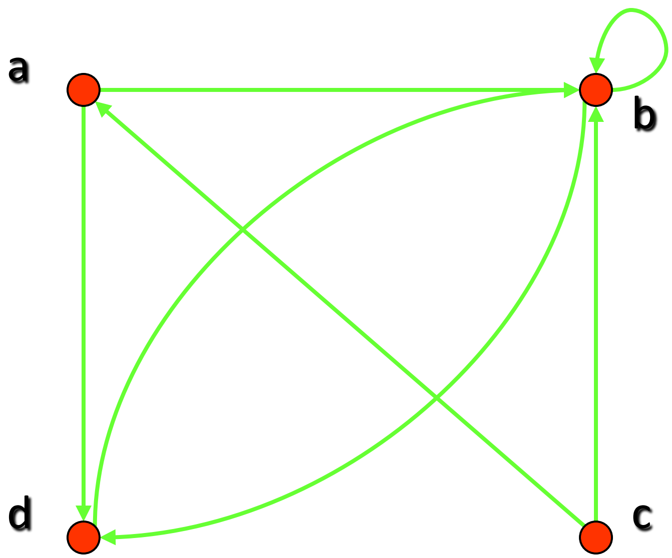
What are the in-degrees and out-degrees of the vertices a, b, c, d in this graph:

$\deg^-(a) = 1$

$\deg^+(a) = 2$

$\deg^-(b) = 4$

$\deg^+(b) = 2$



$\deg^-(d) = 2$

$\deg^+(d) = 1$

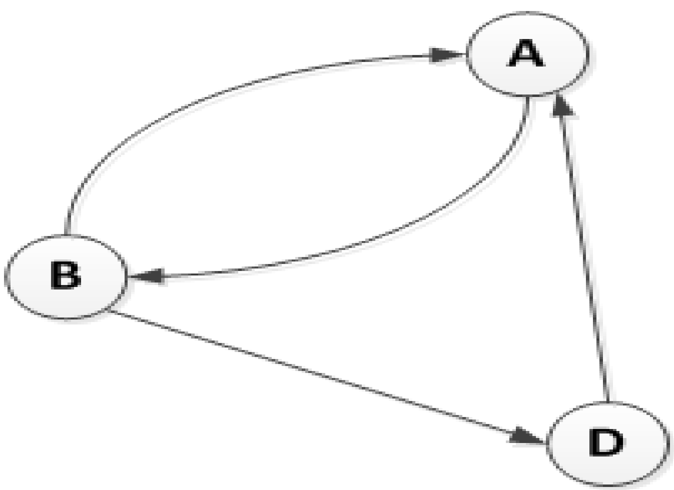
$\deg^-(c) = 0$

$\deg^+(c) = 2$

Handsh. Theorem for Directed Graphs

Theorem: Let G = (V, E) be a graph with directed edges. Then:

$$\sum_{v \in V} \deg^{-}(v) = \sum_{v \in V} \deg^{+}(v) = |E|$$

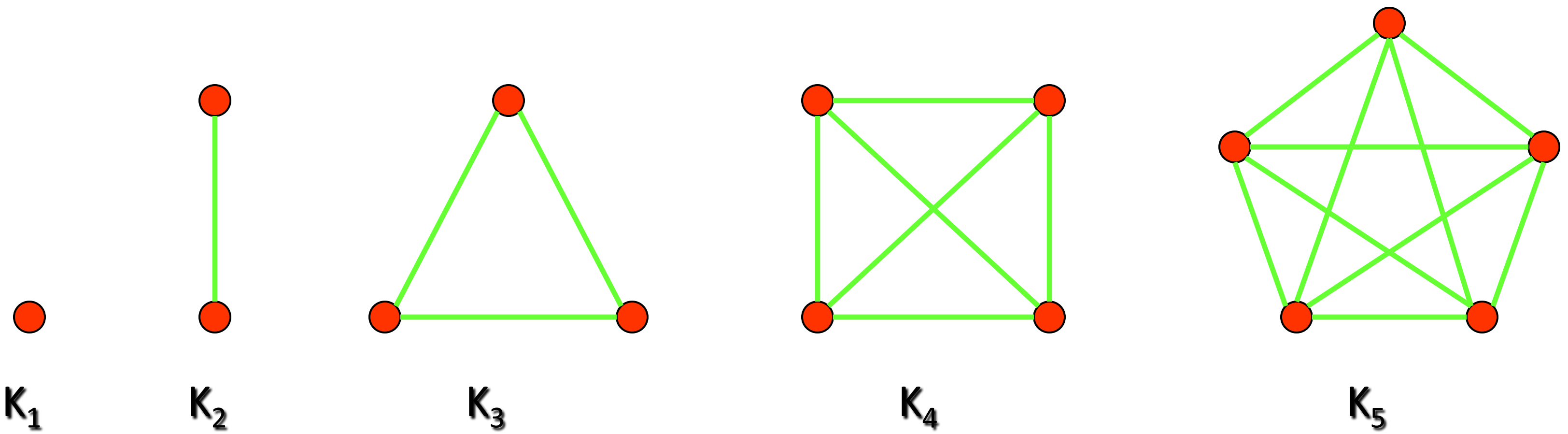


$$\sum_{v \in V} deg^{-}(v) = ?$$

$$\sum_{v \in V} deg^{+}(v) = ?$$

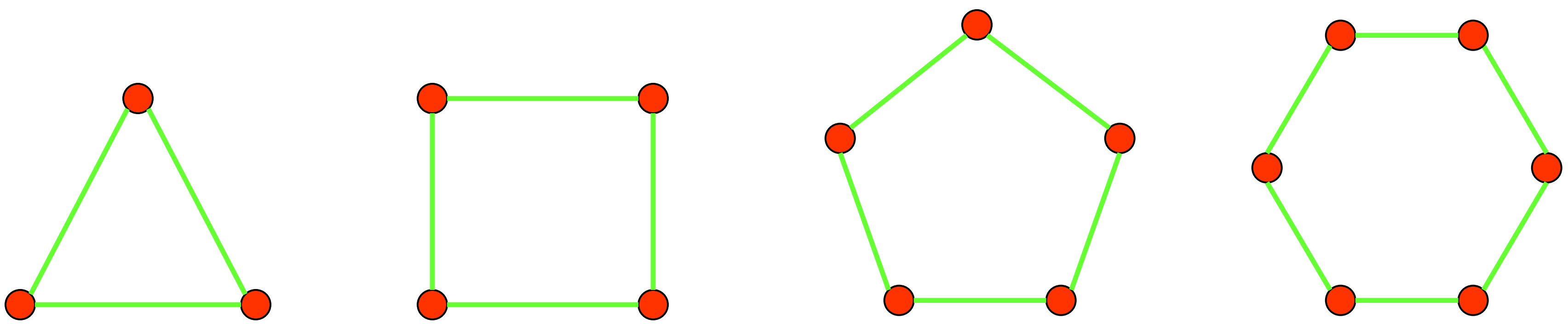
Special Graphs

Definition: The **complete graph** on *n* vertices, denoted by K_n, is the simple graph that contains exactly one edge between each pair of distinct vertices.



Special Graphs

Definition: The **cycle** C_n, $n \geq 3$, consists of n vertices $v_1, v_2, \ldots, v_n$ and edges $\{v_1, v_2\}, \{v_2, v_3\}, \ldots, \{v_{n-1}, v_n\}, \{v_n, v_1\}$.



C_3 C_4 C_5 C_6

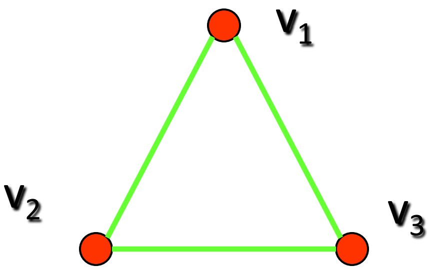
Special Graphs

Definition: A simple graph is called **bipartite** if its vertex set V can be partitioned into two disjoint **nonempty** sets V_1 and V_2 such that **every edge** in the graph connects a vertex in V_1 with a vertex in V_2 (so that no edge in G connects either two vertices in V_1 or two vertices in V_2).

- For example, consider a graph that represents each person in a village by a vertex and each marriage by an edge.
- This graph is **bipartite**, because each edge connects a vertex in the **subset of males** with a vertex in the **subset of females** (if we think of traditional marriages).

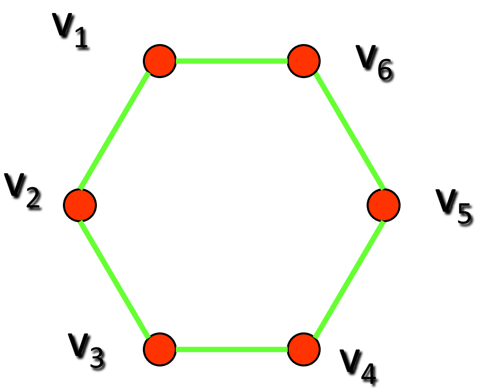
Special Graphs

Example I: Is C_3 bipartite?

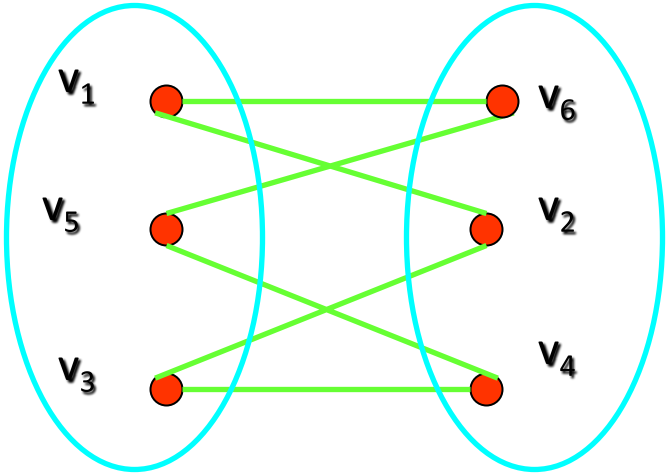


No, because there is no way to partition the vertices into two sets so that there are no edges with both endpoints in the same set.

Example II: Is C_6 bipartite?



Yes, because we can display C_6 like this:



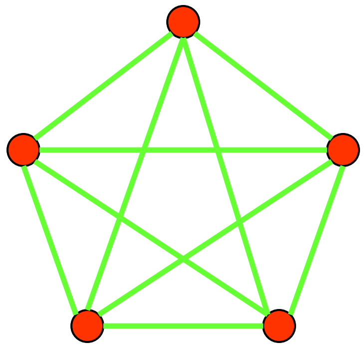
Subgraphs

Definition: A **subgraph** of a graph G = (V, E) is a graph H = (W, F) where $W \subseteq V$ and $F \subseteq E$.

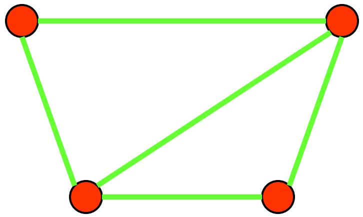
❖ Graph H is a **proper subgraph** of G, if $H \neq G$.

✓ **Note:** Of course, H is a valid graph, so we cannot remove any endpoints of remaining edges when creating H.

Example:



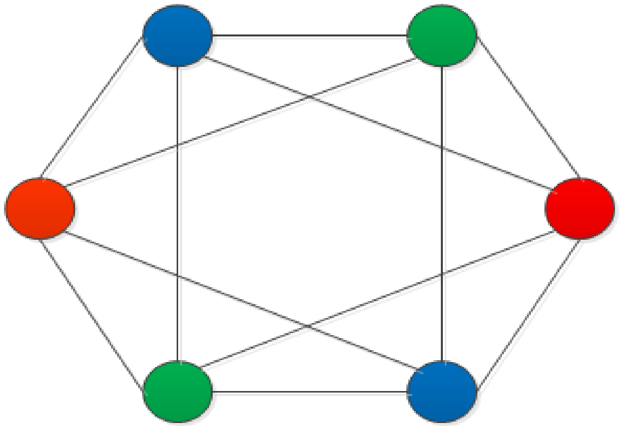
K_5



subgraph of K_5

Graph Coloring

- A **coloring of a graph** is the assignment of a color to each vertex so that no two adjacent vertices are assigned the same color.
- A graph is **k-colorable** if it is possible to color it using *k* colors.



- e.g., graph on right is 3-colorable
- Is it also 2-colorable?

- The **chromatic number** of a graph is the least number of colors needed to color it.
- What is the chromatic number of the above graph?

Question in class

Consider a graph G with vertices {v1, v2, v3, v4} and edges {v1, v2}, {v1, v3}, {v2, v3}, {v2, v4}. Which of the following are valid colorings for G?

1. v1 = red, v2 = green, v3 = blue
2. v1 = red, v2= green, v3 = blue, v4 = red
3. v1 = red, v2= green, v3 = red, v4 = blue

Applications of Graph Coloring

- ✓ Graph coloring has lots of applications, particularly in scheduling.

Example: The math department has 6 committees C1, . . . , C6 that meet once a month.

The committee members are:

C1 = {Allen, Brooks, Marg}
C2 = {Brooks, Jones, Morton}
C3 = {Allen, Marg, Morton}

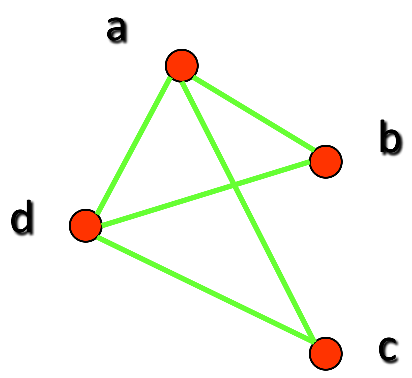
C4 = {Jones, Marg, Morton}
C5 = {Allen, Brooks}
C6 = {Brooks, Marg, Morton}

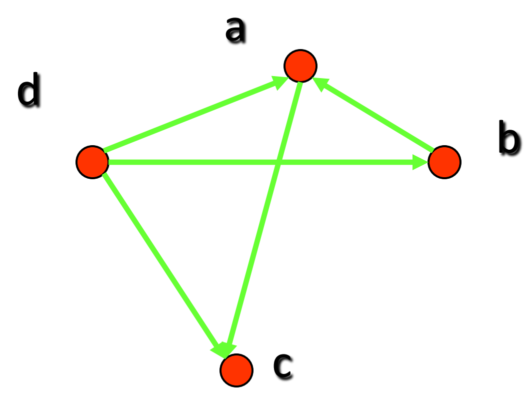
How many different meeting times must be used to guarantee that no one has conflicting meetings?

Graphs and Colorability

- A graph G = (V ,E) is bipartite if and only if it is 2-colorable!!
- Any complete graph K_n has chromatic number n.
- A a simple graph G is always (max_degree(G) + 1)-colorable..

Representing Graphs





Vertex	Adjacent Vertices
a	b, c, d
b	a, d
c	a, d
d	a, b, c

Initial Vertex	Terminal Vertices
a	c
b	a
c	
d	a, b, c

Representing Graphs

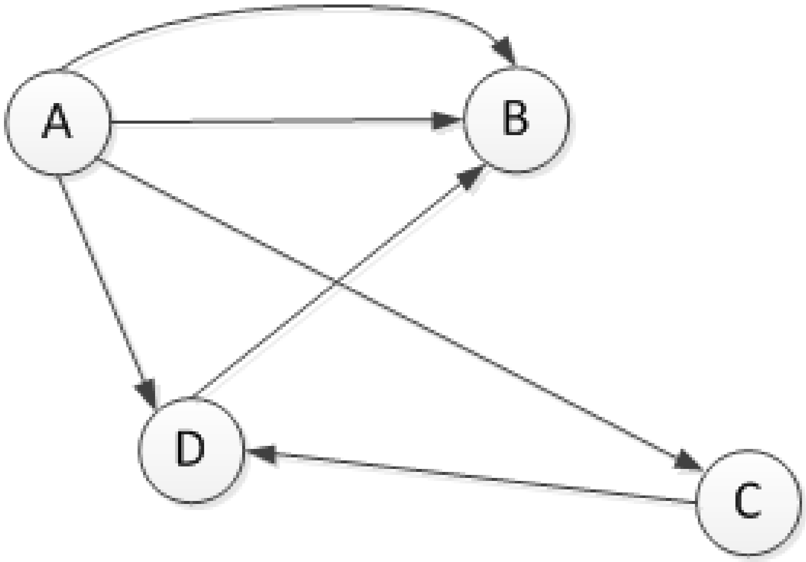
Definition: Let G = (V, E) be a **directed** graph with |V| = n. Suppose that the vertices of G are listed in order as $v_1, v_2, \ldots, v_n$.

❖ The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ matrix $A = [a_{ij}]$ such that:

- a_{ij} = the number of arrows from v_i to v_j for all i,j = 1, 2, .., n.

Representing Graphs

Example: What is the adjacency matrix A_G for the following directed graph G based on the order of vertices a, b, c, d ?



$$A_G = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}$$

Representing Graphs

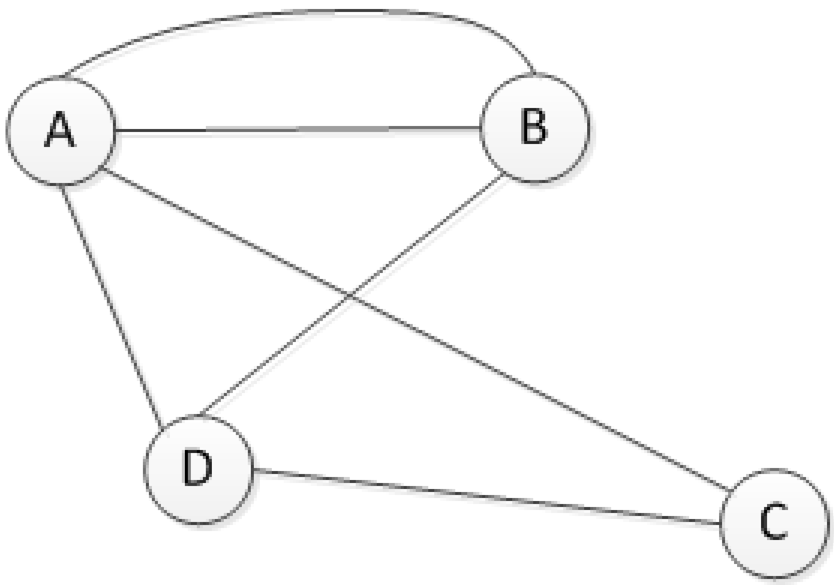
Definition: Let G = (V, E) be an **undirected** graph with |V| = n. Suppose that the vertices of G are listed in order as $v_1, v_2, \ldots, v_n$.

- ❖ The **adjacency matrix** A (or A_G) of G, with respect to this listing of the vertices, is the $n \times n$ matrix $A = [a_{ij}]$ such that:

- a_{ij} = the number of edges connecting $v_i\ and\ v_j$ for all i,j = 1, 2, .., n.

Representing Graphs

Example: What is the adjacency matrix A_G for the following undirected graph G based on the order of vertices a, b, c, d ?



$$A_G = \begin{bmatrix} 0 & 2 & 1 & 1 \\ 2 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix}$$

Note: Adjacency matrices of undirected graphs are always symmetric.

Connectivity

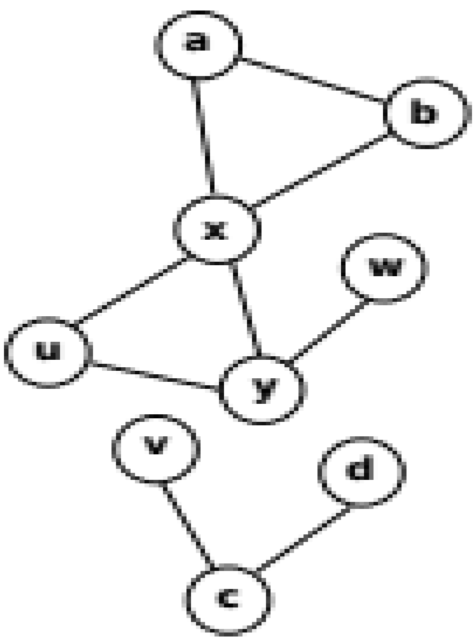
Definition: A **path** of length n from u to v, where n is a positive integer, in an **undirected graph** is a sequence of edges $e_1, e_2, \ldots, e_n$ of the graph such that $e_1 = \{x_0, x_1\}$, $e_2 = \{x_1, x_2\}$, …, $e_n = \{x_{n-1}, x_n\}$, where $x_0 = u$ and $x_n = v$.

- ✓ When the graph is simple, we denote this path by its **vertex sequence** $x_0, x_1, \ldots, x_n$, since it uniquely determines the path.
- ✓ The path is a **circuit** if it begins and ends at the same vertex, that is, if u = v.

Connectivity

Definition (continued): The path or circuit is said to **pass through** or traverse x_1, x_2, …, x_{n-1}.

- A path or circuit is **simple** if it does not contain the same edge more than once.



✓ u, x , y, x , u and u, x , y, u are both **circuits**

✓ u, x , y, u is a **simple** circuit, but u, x , y, x , u is not

✓ **Length of a circuit** is the number of edges it contains, e.g., length of u, x , y, u is 3

Connectivity

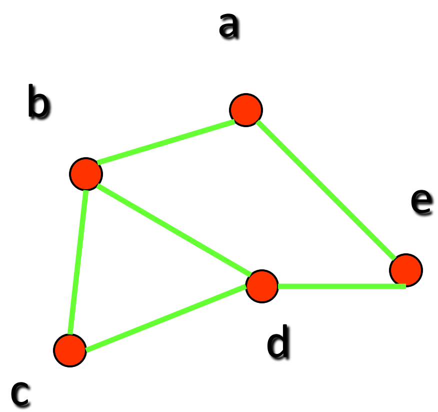
Definition: An undirected graph is called **connected** if there is a path between every pair of distinct vertices in the graph.

- For example, any two computers in a network can communicate if and only if the graph of this network is connected.

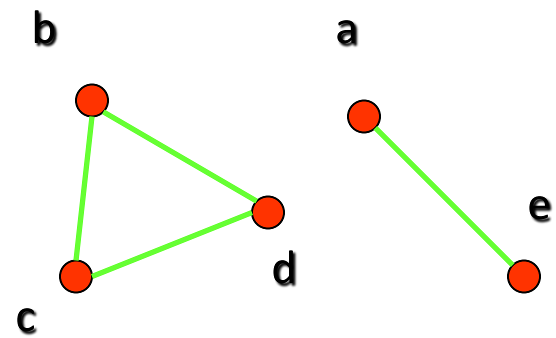
Note: A graph consisting of only one vertex is always connected, because it does not contain any pair of distinct vertices.

Connectivity Examples

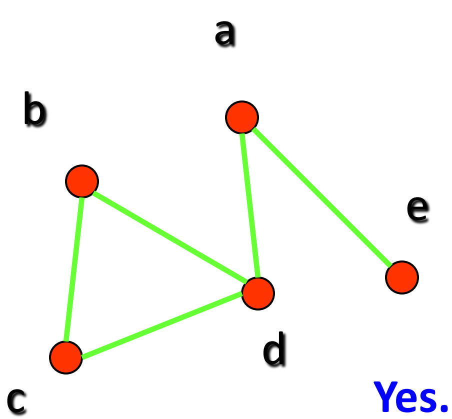
Are the following graphs connected?



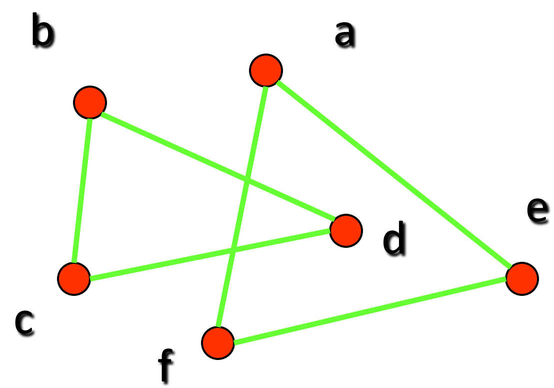
Yes.



No.



Yes.



No.

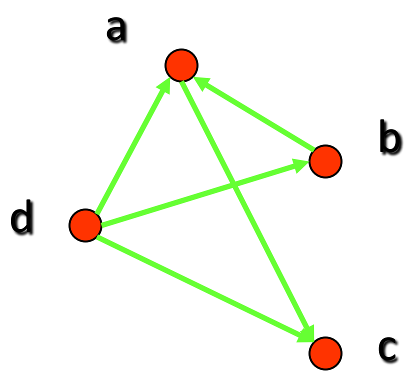
Connectivity

Definition: A **directed** graph is **strongly connected** if there is a path from a to b and from b to a whenever a and b are vertices in the graph.

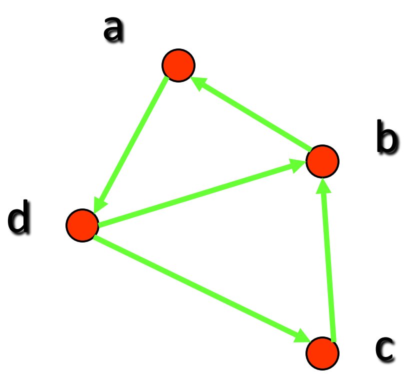
Definition: A **directed** graph is **weakly connected** if there is a path between any two vertices in the underlying undirected graph.

Connectivity

Example: Are the following directed graphs strongly or weakly connected?



Weakly connected, because, for example, there is no path from b to d.

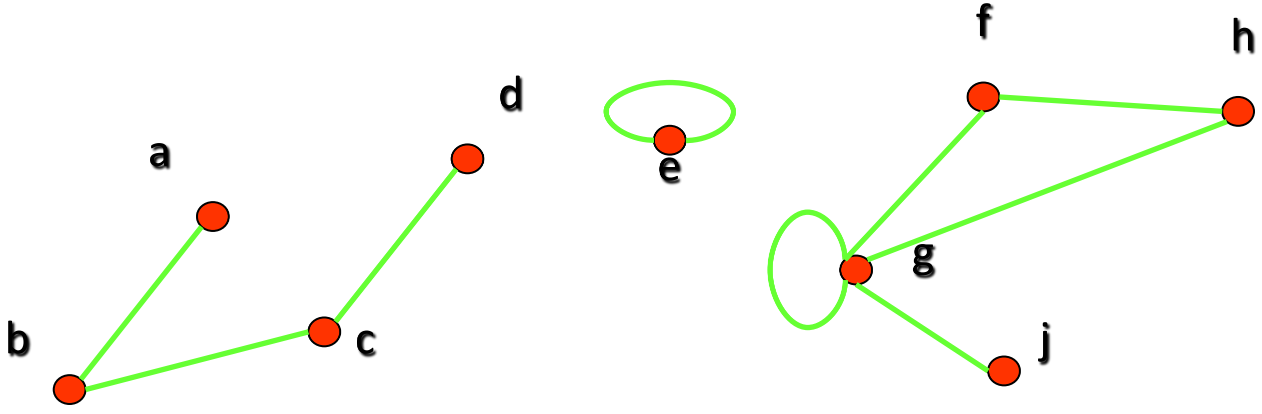


Strongly connected, because there are paths between all possible pairs of vertices.

Connected Components

Definition: A graph that is not connected is the union of two or more connected sub-graphs, each pair of which has no vertex in common. These disjoint connected sub-graphs are called the **connected components** of the graph.

Example: What are the connected components in the following graph?



Solution: The connected components are the graphs with vertices {a, b, c, d}, {e}, {f, g, h, j}.