Lecture 1

- Introduction to:
  - The language of Sets
  - The language of Relations
  - The language of Functions

Reading (Epp's textbook)
1.1-1.3, 6.1

Variables

A variable can be used as a placeholder when you want to talk about something but either:

1) You imagine that it has one or more values but you don't know what they are
   - Is there a number with the following property: doubling it and adding 4 gives the same result as squaring it?
   - Is there a number x with the property that $2x + 4 = x^2$?
2) You want whatever about it to be equally true for all elements in a given set
   - No matter what number might be chosen, if it is greater than 2, then its square is greater than 4.
   - No matter what number n might be chosen, if n is greater than 2, then n^2 is greater than 4.

Sets

- A set is a collection of elements.
- If S is a set,
  - The notation $x \in S$ means that x is an element of S.
  - The notation $x \notin S$ means that x is not an element of S.
- There is only one set with no elements, named the **empty set** and denoted by the symbol ∅.
- Set-roster notation, e.g. {1, 2, 5} denotes the set whose elements are 1, 2, and 5.
- Examples:
  - Let A = {1, 2, 3, 4}, B = { 4, 2, 1, 3}, and C = {4, 4, 1, 1, 1, 3, 2, 2}. What are the elements of A, B, and C? How are A, B, and C related?
  - Is {3} = 3?
  - How many elements are in the set { 1, {1}, 2, {2}, {1, 2}}?

Set-Builder Notation

We define a new set to be **the set of all elements x in S such that $P(x)$ (a property that elements of S may or may not satisfy) is true.**

$$\{x \in S \mid P(x)\}$$

such that

- Examples:
  - *a.* $\{x \in R \mid -4 < x < 3\}$

    The open interval of real numbers between -4 and 3 (continuous).
  - *b.* $\{x \in Z \mid -3 < x \leq 2\}$

    $\{-2, -1, 0, 1, 2\}$ (Discrete).
  - *c.* $\{x \in Z^{-} \mid -4 < x \leq 100\}$

    $\{-3, -2, -1\}$ (Discrete).

Subsets

- If *A* and *B* are sets, then *A* is called as subset of *B*, written $A \subseteq B$, if, and only if, every element of *A* is also an element of *B*.
  - $A \subseteq B$ means that For all elements x, if $x \in A$ then $x \in B$.
  - $A \nsubseteq B$ means that There is at least one element x such that $x \in A$ and $x \notin B$.
- Let *A* and *B* be sets. *A* is a proper subset ($\subset$) of *B*, if, and only if, ($\Leftrightarrow$) 1) every element of *A* is in *B* ($A \subseteq B$), 2) but there is at least one element of *B* that is not in *A*.
- Which of the following are true statements?

  a. $\{x \in Z \mid 0 \leq x \leq 100\} \subseteq Z^+$

  b. $\{100, 200, 300\} \subset Z^+$

  c. $\{100, 200, 300\} \subset \{100, 200, 300\}$

  d. $\{2\} \subseteq \{1, 2, 3\}$

  e. $\{2\} \subset \{\{1\}, \{2\}\}$

Only (b), and (d) are true.

Subsets: Proof and Disproof

- *Element Argument:*

Let sets X and Y be given. To prove that $X \subseteq Y$,

1. suppose that x is a particular but arbitrarily chosen element of X.
2. show that x is an element of Y.

- Example:

$$A = \{m \in Z^+ \mid m = 4r + 12 \; for \; some \; r \in Z^+\}$$
$$B = \{n \in Z^+ \mid n = 4s \; for \; some \; s \in Z^+\}$$

- Prove that $A \subseteq B$.
  1. Suppose x is a particular but arbitrarily chosen element of A.
  2. We must show that $x \in B$. By definition of B, this means we must show that $x = 4 \times (some \; integer \in Z^+)$.
     i. $x = 4r + 12 = 4 \times (r + 3)$.
     ii. Let $s = r + 3$. Does $s \in Z^+$?
     iii. Now we must check that $x = 4 \times s$.

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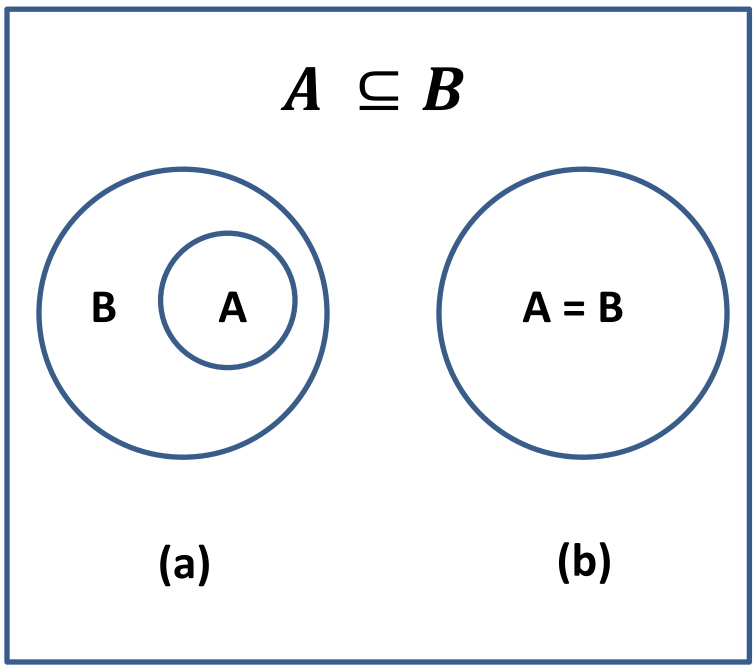
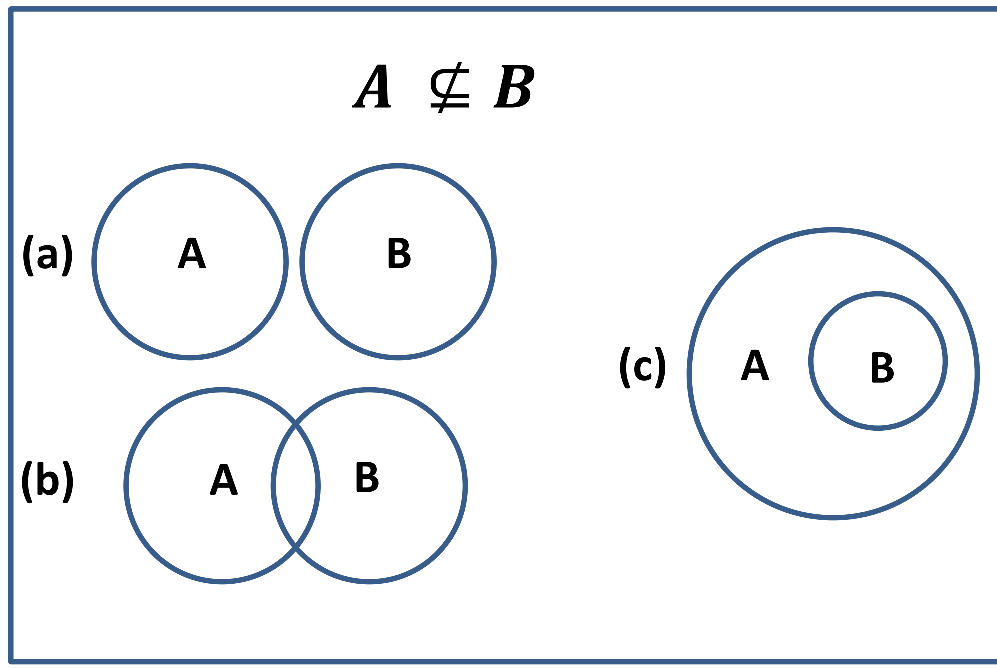
- Disprove that $\mathrm{B} \subseteq A$.
  - To disprove a statement means to show that it is false.
  - You must find an element of B that is not an element of A.
  - Let $x = 4$. Then $x \in B$ but $x \notin A$ because there is no (positive integer) $\mathrm{r} \in Z^+$ such that $4 = 4r + 12$.

Set Equality – Venn Diagrams

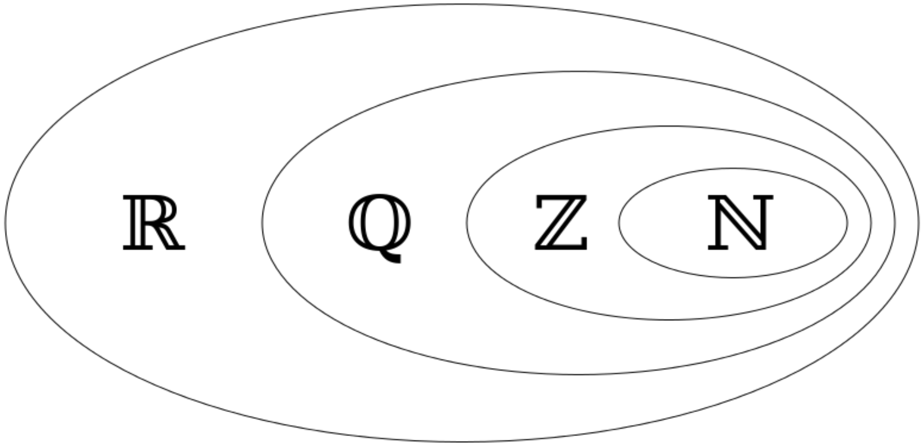
- Given sets A and B, A equals B, written $A = B$, if, and only if, every element of A is in B and every element of B is in A.

Symbolically: $A = B \Leftrightarrow A \subseteq B$ and $B \subseteq A$





Relations among Sets of Numbers



Symbol	Set
R	Set of real numbers
Q	Set of all rational numbers
Z	Set of all integers
N	Set of all natural numbers (Z^{nonneg} or Z^{+})

Operations on Sets

Let sets A and B be subsets of a universal set U.

1. The **union** of A and B, denoted $\boldsymbol{A \cup B}$, is the set of all elements that are in at least one of A or B.

   Symbolically: $A \cup B = \{x \in U \mid x \in A \text{ } or \text{ } x \in B\}$.

2. The **intersection** of A and B, denoted $\boldsymbol{A \cap B}$, is the set of all elements that are common to both A and B.

   Symbolically: $A \cap B = \{x \in U \mid x \in A \text{ } and \text{ } x \in B\}$.

3. The **difference** of B minus A, denoted $\boldsymbol{B - A}$, is the set of all elements that are in B and not A.

   Symbolically: $B - A = \{x \in U \mid x \in B \text{ } and \text{ } x \notin A\}$.

4. The **complement** of A, denoted A^c, is the set of all elements in U that are not in A.

   Symbolically: $A^c = \{x \in U \mid x \notin A\}$.

Partitions of Sets

- Two sets are called **disjoint** if, and only if, they have no elements in common.

  Symbolically: A and B are disjoint ⇔ $A \cap B = \emptyset$.

  - Are $A = \{1, 3, 5, 6\}$ and $B = \{2, 4, 6\}$ disjoint?

- Sets A_1, A_2, A_3 … are **mutually disjoint** if, and only if, no two sets A_i and A_j with distinct subscripts have any element in common. More precisely, for all *i, j* = 1, 2, 3, …

$$A_i \cap A_j = \emptyset \text{ whenever } i \neq j.$$

- A finite or infinite collection of nonempty sets $\{A_1, A_2, A_3 \dots\}$ is a **partition** of set A if, and only if,

  1. A is the union of all the A_i
  2. The sets A_1, A_2, A_3 … are mutually disjoint.

Let $A = \{1, 2, 3, 4, 5, 6\}, A_1 = \{1, 2\}, A_2 = \{3, 4, 5\}$ and $A_3 = \{5, 6\}$.
Is $\{A_1, A_2, A_3\}$ a partition of A?

Power Sets

- P(A) "power set of A"
- P(A) = {B | B ⊆ A} (contains all subsets of A)
- Examples:
  - A = {x, y, z}

    P(A) = {∅, {x}, {y}, {z}, {x, y}, {x, z}, {y, z}, {x, y, z}}
  - A = ∅

    P(A) = {∅}
- If a set S contains n distinct elements, $n \in \boldsymbol{N}$, we call S a finite set with cardinality $|S| = n$.
- Cardinality of power sets:
  - $|P(A)| = 2^{|A|}$

Cartesian Products

- The symbol (a, b) denotes the ordered pair (ordered two-tuple) consisting of a and b together with the specification that
  - a is the first element of the pair
  - b is the second element of the pair.
- $(a, b) = (c, d)$ means that $a = c$ and $b = d$. (Is (1, 2)=(2, 1) ?)
- In general two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.
- **Cartesian product of two sets A and B**:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

- Example: A = {good, bad}, B = {student, prof}

$$A \times B = \{(good, student), (good, prof), (bad, student), (bad, prof)\}.$$

Is $A \times B$ = B $\times$ A ?

Cartesian Products

Note that:

- $A \times \varnothing = \varnothing$
- $\varnothing \times A = \varnothing$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$

➢ **Cartesian product of two or more sets is defined as**:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \ldots a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Relations

- Let A and B be sets. **A relation $\boldsymbol{R}$ from $\boldsymbol{A}$ to $\boldsymbol{B}$** is a subset of $A\ x\ B$. Given and ordered pair (x, y) in $A\ x\ B$, **$\boldsymbol{x}$ is related to $\boldsymbol{y}$ by $\boldsymbol{R}$**, written $x\ R\ y$, if, and only if, (x, y) is in R.
- The set A is called the **domain** of R and the set B is called its **co-domain**.
- Example: Let $A = \{2, 4, 6\}$ and $B = \{2, 3\}$. Given any $(x, y) \in A \times B$,
  - a relation R from A to B is defined as follows:

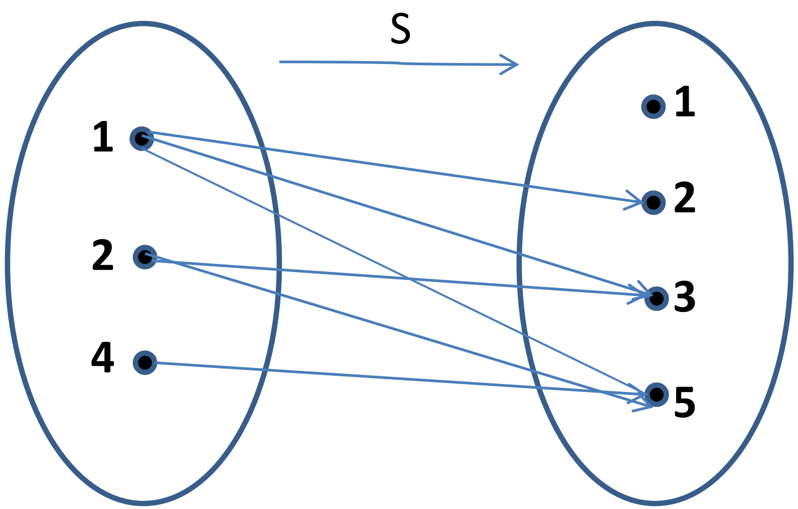
    $(x, y) \in R$ means that $\frac{x}{y}$ is an integer.

    - Which ordered pairs are in R? R = {(2, 2), (4, 2), (6, 2), (6,3)}
    - Is $2\ R\ 3$? Is $4\ R\ 2$? Is $6\ R\ 3$?

Arrow Diagram of a Relation

- Let $A = \{1, 2, 4\}$ and $B = \{1, 2, 3, 5\}$ and define relation S from A to B as follows:
  - For all $(x, y) \in A \times B$,

    $(x, y) \in S$ means that x < y



Functions

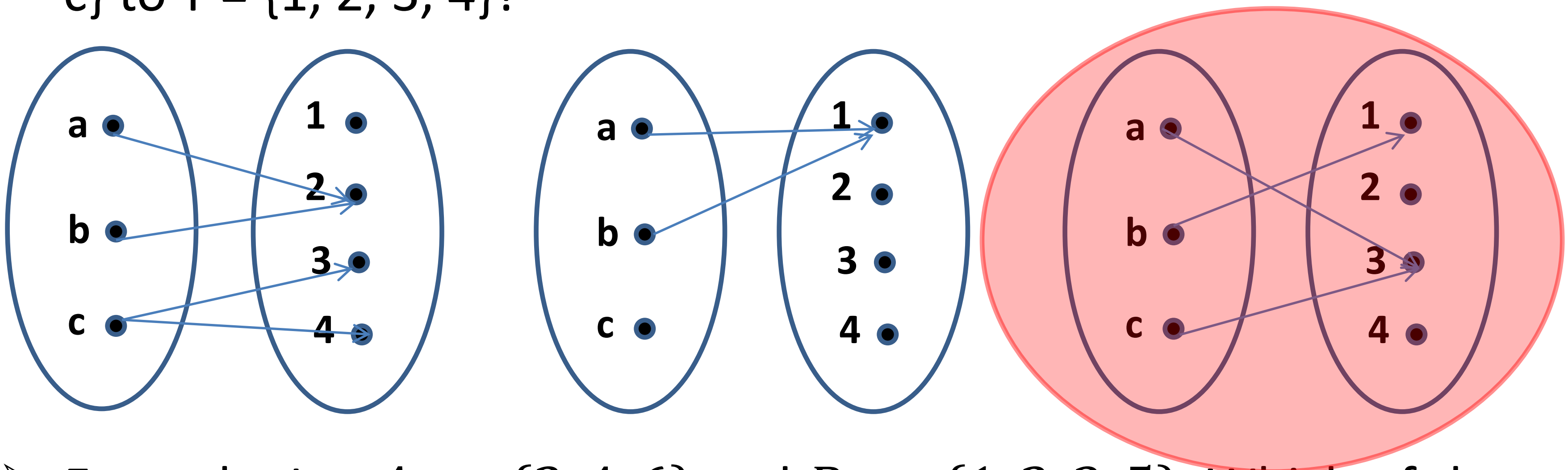
- **A function $\boldsymbol{F}$ from a set $\boldsymbol{A}$ to a set $\boldsymbol{B}$**, denoted $f: A \to B$, is a relation with **domain** A and **co-domain** B that satisfies the following two properties:
  1. For **every** element x in A, there is an element y in B such that $(x, y) \in F$.
  2. For all elements x in A and y and z in B,

     If $(x, y) \in F$ and $(x, z) \in F$, then y = z.

- **range of $\boldsymbol{f}$** $= \{y \in B \mid y = f(x), for\ some\ x\ in\ A\}$.
- **the inverse image of $\boldsymbol{y}$** $= \{x \in A \mid f(x) = y\}$.

Functions Arrow Diagrams

- Which of the arrow diagrams define functions from X = {a, b, c} to Y = {1, 2, 3, 4}?



- Example: Let $A = \{2, 4, 6\}$ and $B = \{1, 2, 3, 5\}$. Which of the relations R, and S defined below are functions from A to B?

  a. $R = \{(2,5), (4,1), (4,3), (6,2)\}$

  b. For all $(x, y) \in A \times B$, $(x, y) \in S$ means that $y = \frac{x}{2}$

Only B.

$S = \{ (2,1), (4,2), (6,3) \}$

Equality of Functions

- If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.
- Example: Let $J_3 = \{0, 2, 4\}$, $J_4 = \{0, 1\}$, and define functions f and g from J_3 to J_4 , as follows: For all x in J_3,

  $$f(x) = (x^2 + x + 1)\ mod\ 3 \text{ and } g(x) = (x + 2)^2\ mod\ 3$$

  Does $f = g$?

x	$x^2 + x + 1$	$(x^2 + x + 1)\ mod\ 3$	$(x + 2)^2$	$(x + 2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$
4	21	$21\ mod\ 3 = 0$	36	$36\ mod\ 3 = 0$

Yes, the table of values shows that $f(x) = g(x)$ for all $x \in J_3$.