Counting

- Introduction
- Possibility Trees
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- Counting Elements of Disjoint Sets
- The Pigeonhole Principle
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Reading (Epp's textbook)
9.1-9.5

Introduction

- Imagine tossing two coins and observing whether 0, 1, or 2 heads are obtained.
- It would be natural to guess that each of these events occurs about one-third of the time.
- But in fact this is not the case!!!

Table 9.1.1 Experimental Data Obtained from Tossing Two Quarters 50 Times

Event	Tally	Frequency (Number of times the event occurred)	Relative Frequency (Fraction of times the event occurred)
2 heads obtained	卌 卌 |	11	22%
1 head obtained	卌 卌 卌 卌 卌 ||	27	54%
0 heads obtained	卌 卌 ||	12	24%

Introduction

- Call the two coins *A* and *B*, and suppose that each is perfectly balanced (random process).
- Then each has an equal chance of coming up heads (50%) or tails (50%), and when the two are tossed together, the four outcomes pictured in Figure 9.1.2 are all equally likely.

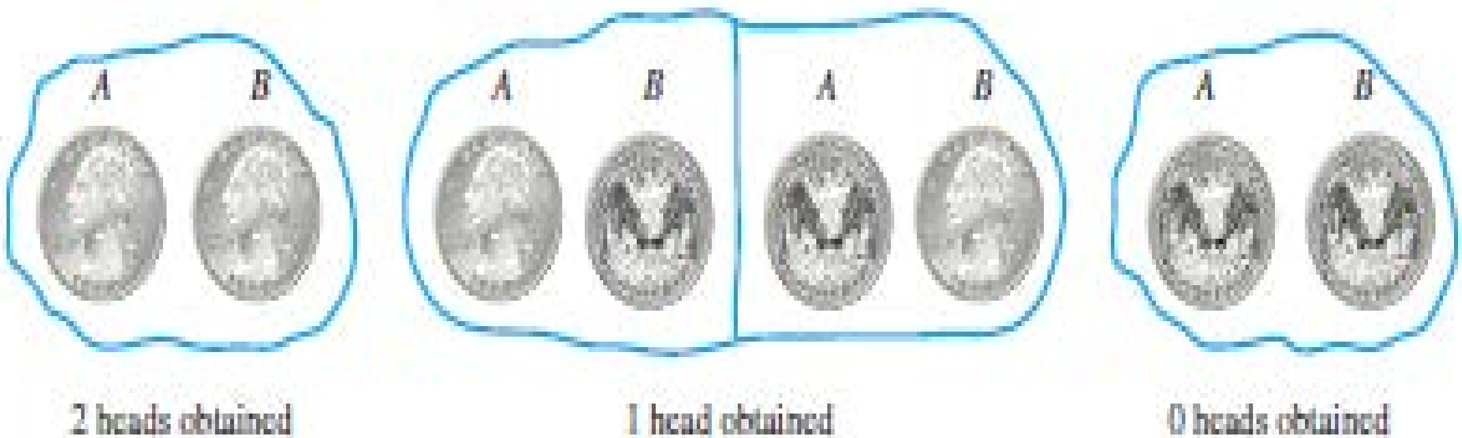


Figure 9.1.2 Equally Likely Outcomes from Tossing Two Balanced Coins

Introduction -Definitions

- A **sample space** is the set of all possible outcomes of a random process or experiment.
- An **event** is a subset of a sample space.

Example:

- What is the sample space of tossing two coins?

  Coin_A = { H, T} , Coin_B = {H, T}

  Coin_A x Coin_B = { (H, H), (H, T), (T, H), (T, T) } is the sample space.

- Which are the possible events?

  Obtain 0, 1, or 2 heads.

Probability Formula

If S is a finite sample space in which all outcomes are equally likely and E is an event in S, then the ***probability of E***, denoted ***P*(*E*)**, is:

$$P(E) = \frac{the\ number\ of\ outcomes\ in\ E}{the\ total\ number\ of\ outcomes\ in\ S}$$

Example:

- What is the probability of obtaining 1 head if we toss two coins?

$$P(obtain\ 1\ head) = \frac{|\{(H,T),(T,H)\}|}{|\{(T,T),(H,T),(T,H),(H,H)\}|} = \frac{1}{2}$$

- What is the probability of obtaining 2 heads?

Another example – in class

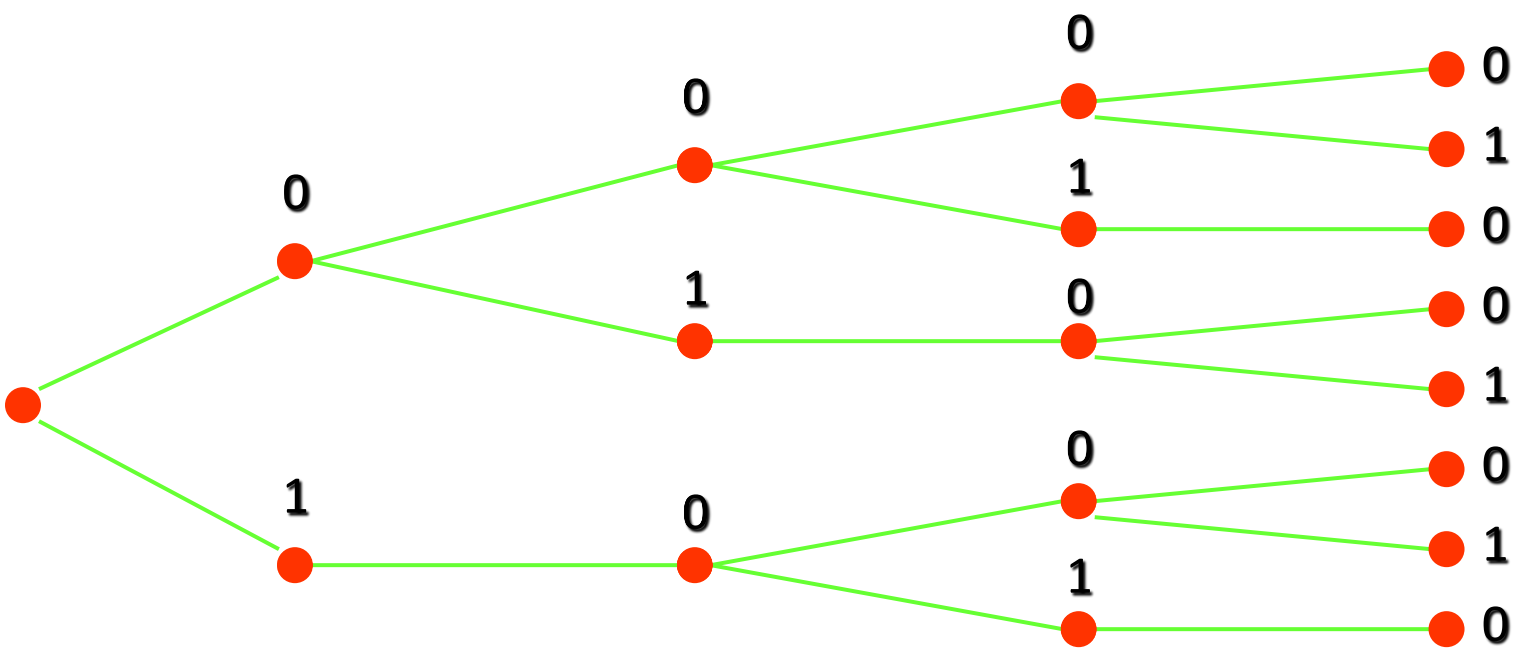
A die is one of a pair of dice. It is a cube with six sides, each containing from one to six dots. Suppose a pair of dice are rolled together, and the numbers of dots that occur face up on each are recorded.

- What is the sample space of rolling a pair of dice?
- What is the cardinality of the sample space?
- Use set notation to write the event *E* that the numbers showing face up have a sum of 5 and find the probability of this event.
- Which event (sum of the numbers showing face up) has the highest probability?

Possibility Trees - Example

How many bit strings of length four do not have two consecutive 1s?

Step 1 (1st bit) | Step 2 (2nd bit) | Step 3 (3rd bit) | Step 4 (4th bit)



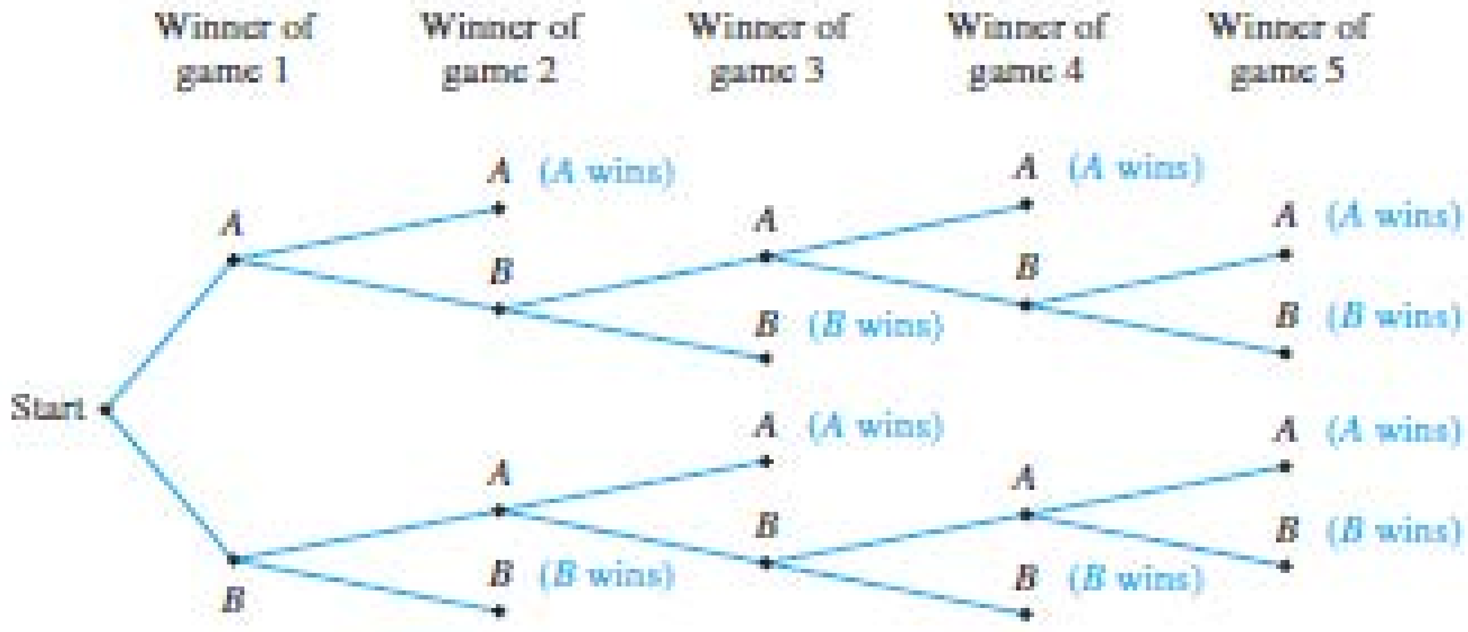
There are 8 strings.

Another Example – in class

Teams *A* and *B* are to play each other repeatedly until one wins two games in a row or a total of three games.

- ✓ One way in which this tournament can be played is for *A* to win the first game, *B* to win the second, and *A* to win the third and fourth games.

- How many ways can the tournament be played?



The Multiplication Rule

If an operation consists of k steps and

the first step can be performed in $n1$ ways,

the second step can be performed in $n2$ ways [***regardless of how the first step was performed***],

.
.
.

the kth step can be performed in nk ways [***regardless of how the preceding steps were performed***],

then the entire operation can be performed in

$n1 \times n2 \times \cdots \times \; nk$ ways.

Multiplication Rule-Example

A typical PIN (personal identification number) is a sequence of any four symbols chosen from the 26 letters in the alphabet and the ten digits, with repetition allowed.

- How many different PINs are possible?

  Step 1: Choose the first symbol. (36 choices)

  Step 2: Choose the second symbol. (36 choices)

  Step 3: Choose the third symbol. (36 choices)

  Step 4: Choose the fourth symbol. (36 choices)

By the multiplication rule, there are $36 \cdot 36 \cdot 36 \cdot 36 = 36^4 =$ 1,679,616 PINs in all.

Multiplication Rule-Example

- Now suppose that repetition is not allowed. How many different PINs are there?

  Step 1: Choose the first symbol. (36 choices)

  Step 2: Choose the second symbol. (35 choices)

  Step 3: Choose the third symbol. (34 choices)

  Step 4: Choose the fourth symbol. (33 choices)

By the multiplication rule, there are $36 \cdot 35 \cdot 34 \cdot 33 =$ 1,413,720 PINs with no repeated symbol.

- If all PINs are equally likely, what is the probability that a PIN chosen at random contains no repeated symbol?

  The probability that a PIN chosen at random contains no repeated symbol is 1,413,720 / 1,679,616 ≈ .8417. (84%)

Example - in class

How many bit strings of length 8 end with 00?

Solution:

There are two ways to pick the first bit (0 or 1),
two ways to pick the second bit (0 or 1),
.
.
.
two ways to pick the sixth bit (0 or 1),
one way to pick the seventh bit (0), and
one way to pick the eighth bit (0).

Multiplication rule: Can be done in $2^6 = 64$ ways.

Addition Rule

- Two basic very useful decomposition rules:

  1. Multiplication rule
  2. **Addition rule**

- Suppose a task A can be done either in way B or in way C

- Suppose there are n1 ways to do B, and n2 ways to do C

- Addition rule: There are n1 + n2 ways to do A.

Addition Rule - Example

Suppose either a CS faculty or CS student must be chosen as representative for a committee.
There are 14 faculty, and 50 students.

- How many ways are there to choose the representative?

By the addition rule, 50 + 14 = 64 ways.

Note: Just like the multiplication rule, the addition rule can be extended to more than two tasks.

Addition Rule - Example

A student can choose a senior project from one of three lists. First list contains 23 projects; second list has 15 projects, and third has 19 projects. Also, no project appears on more than one list.

- How many different projects can a student choose?

By addition rule, 23 + 15 + 19 = 57.

What if some of the projects appeared on both lists?

Caveat: For addition rule to apply, the possibilities must be mutually exclusive.

More Complex Counting Problems

Problems so far required either only multiplication or only addition rule.

But more complex problems require a combination of both!

Recall: Addition rule only applies if a task is as disjunction of two or more mutually exclusive tasks.

What do we do if the tasks aren’t mutually exclusive?

Generalization of the addition rule: **inclusion-exclusion principle**

Inclusion - Exclusion

Example:

How many bit strings of length 8 either start with a 1 or end with 00?

Task 1: Construct a string of length 8 that starts with a 1.

There is one way to pick the first bit (1),
two ways to pick the second bit (0 or 1),
two ways to pick the third bit (0 or 1),
.
.
.
two ways to pick the eighth bit (0 or 1).

Multiplication rule: Task 1 can be done in $1 \cdot 2^7 = 128$ ways.

Inclusion - Exclusion

Task 2: Construct a string of length 8 that ends with 00.

There are two ways to pick the first bit (0 or 1),
two ways to pick the second bit (0 or 1),
.
.
.
two ways to pick the sixth bit (0 or 1),
one way to pick the seventh bit (0), and
one way to pick the eighth bit (0).

Multiplication rule: Task 2 can be done in $2^6 = 64$ ways.

Since there are 128 ways to do Task 1 and 64 ways to do Task 2, does this mean that there are 192 bit strings either starting with 1 or ending with 00 ? (!!!)

Inclusion - Exclusion

- When we carry out Task 1 and create strings starting with 1, some of these strings end with 00.
- We have to subtract the cases when Tasks 1 and 2 are done at the same time.

- How many cases are there, that is, how many strings start with 1 **and** end with 00?

  There is one way to pick the first bit (1),
  two ways for the second, …, sixth bit (0 or 1),
  one way for the seventh, eighth bit (0).

Multiplication rule: In $2^5 = 32$ cases, Tasks 1 and 2 are carried out at the same time. Hence, there are 128 + 64 – 32 = 160 ways to do either task.

Counting Elements of disjoint Sets

The addition and multiplication rules can also be phrased in terms of **set theory**.

Addition rule: Let $A_1, A_2, \ldots, A_m$ be disjoint sets. Then the number of ways to choose any element from one of these sets is:

$$|A_1 \cup A_2 \cup \ldots \cup A_m| = |A_1| + |A_2| + \ldots + |A_m|.$$

Multiplication rule: Let $A_1, A_2, \ldots, A_m$ be finite sets. Then the number of ways to choose one element from each set in the order $A_1, A_2, \ldots, A_m$ is:

$$|A_1 \times A_2 \times \ldots \times A_m| = |A_1| \cdot |A_2| \cdot \ldots \cdot |A_m|.$$

Inclusion/Exclusion Rule for Sets

The Inclusion/Exclusion Rule for Two or Three Sets

If A, B, and C are any finite sets, then

$$|A \cup B| = |A| + |B| - |A \cap B|$$

and

$$|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

Inclusion/Exclusion Example - Bonus

A professor in a discrete mathematics class passes out a form asking students to check all the mathematics and computer science courses they have recently taken. The findings are that out of a total of 50 students in the class,

- 30 took pre-calculus;
- 16 took both pre-calculus and Java;
- 18 took calculus;
- 8 took both calculus and Java;
- 26 took Java;
- 47 took at least one of the three courses.
- 9 took both pre-calculus and calculus;

- How many students took all three courses?

The Pigeonhole Principle

The pigeonhole principle: If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 1: If there are 11 players in a soccer team that wins 12-0, there must be at least one player in the team who scored at least twice.

Example 2: If you have 6 classes from Monday to Friday, there must be at least one day on which you have at least two classes.

The Pigeonhole Principle

The pigeonhole principle: If (k + 1) or more objects are placed into k boxes, then there is at least one box containing two or more of the objects.

Example 3: Consider an event with 367 people. Is it possible no pair of people have the same birthday?

Example 4: Consider function f from a set with k + 1 or more elements to a set with k elements. Is it possible f is one-to-one?

Example 5: Consider n married couples. How many of the 2n people must be selected to guarantee there is at least one married couple?

The Pigeonhole Principle

The generalized pigeonhole principle: If N objects are placed into k boxes, then there is at least one box containing at least $\lceil N/k \rceil$ of the objects.

Example 1: In our 32-student class, at least 7 students will get the same letter grade (A, B, C, D, or F).

Example 2: If there are 30 students in a class, at least how many must be born in the same month?

Example 3: Assume you have a drawer containing a random distribution of a dozen brown socks and a dozen black socks. It is dark, so how many socks do you have to pick to be sure that among them there is a matching pair?

Permutations

A **permutation** of a set of distinct objects is an ordered arrangement of these objects

- ✓ No object can be selected more than once.
- ✓ Order of arrangement matters.

Example: S = {a, b, c}. What are the permutations of S?

How many Permutations?

In general, given a set of *n* objects, how many permutations does the set have?

Imagine forming a permutation as an *n*-step operation:

Step 1: Choose an element to write first. (n choices)

Step 2: Choose an element to write second. (n – 1 choices)

.
.
.

Step *n*: Choose an element to write *n*th. (1 choice)

Multiplication rule: There are $n \times (n-1) \times \cdots \times 2 \times 1 = n!$ permutations.

Permutations Examples – in class

- Consider the set {7, 10, 23, 4}. How many permutations?

- How many permutations {a, b, c, d, e, f, g} end with a?

- How many permutations of letters A, B, C, D, E, F, G contain ”ABC” as a substring?

Permutations - Example

How many ways are there to pick a set of 3 people from a group of 6?

There are 6 choices for the first person, 5 for the second one, and 4 for the third one, so there are $6 \cdot 5 \cdot 4 = 120$ ways to do this.

This is not the correct result!

For example, picking person C, then person A, and then person E leads to the **same group** as first picking E, then C, and then A.

However, these cases are counted **separately** in the above equation.

r-Permutations

- An ***r*-permutation** of a set of *n* elements is an ordered selection of *r* elements taken from the set of *n* elements.
- The number of *r*-permutations of a set of *n* elements is denoted ***P(n, r).***

If *n* and *r* are integers and $1 \le r \le n$, then the number of *r*-permutations of a set of *n* elements is given by the formula:

$$P(n,r) = n(n-1)(n-2)\cdots(n-r+1) = \frac{n!}{(n-r)!}$$

Example:

How many different ways can three of the letters of the word *EXAMPLE* be chosen and written in a row?

r-Permutations Examples – in class

- What is the number of 2-permutations of set {a, b, c, d, e}?

- How many ways to select first-prize winner, second-prize winner, third-prize winner from 10 people in a contest?

- Salesman must visit 4 cities from list of 10 cities: Must begin in Chicago, but can choose the remaining cities and order. How many possible itinerary choices are there?

r-Combinations

- An **r-combination** of elements of a set is an unordered selection of r elements from the set.
- Thus, an r-combination is simply a subset of the set with r elements.

Example: Let S = {1, 2, 3, 4}.

Then {1, 3, 4} is a 3-combination from S.

- The number of r-combinations of a set with n distinct elements is denoted by C(n, r).

Example:

C(4, 2) = 6, since, for example, the 2-combinations of a set {1, 2, 3, 4} are {1, 2}, {1, 3}, {1, 4}, {2, 3}, {2, 4}, {3, 4}.

r-Combinations

How can we calculate C(n, r)?

Consider that we can obtain the r-permutation of a set in the following way:

First, we form all the r-combinations of the set (there are C(n, r) such r-combinations).

Then, we generate all possible orderings in each of these r-combinations (there are P(r, r) such orderings in each case).

Therefore, we have:

$P(n, r) = C(n, r) \cdot P(r, r)$

Combinations

✓ $C(n, r) = P(n, r)/P(r, r)$
$= n!/(n - r)!/(r!/(r - r)!)$
$= n!/(r!(n - r)!)$

Now we can answer our initial question:

How many ways are there to pick a set of 3 people from a group of 6 (disregarding the order of picking)?

$C(6, 3) = 6!/(3!\cdot 3!) = 720/(6\cdot 6) = 720/36 = 20$

There are 20 different ways, that is, 20 different groups to be picked.

Combinations

Corollary:

Let n and r be nonnegative integers with $r \leq n$.
Then $C(n, r) = C(n, n - r)$.

✓ Note that **"picking a group of r people from a group of n people"** is the same as **"splitting a group of n people into a group of r people and another group of (n – r) people".**

Combinations Example – in class

Example:
A soccer club has 8 female and 7 male members. For today's match, the coach wants to have 6 female and 5 male players on the grass. How many possible configurations are there?

$$C(8, 6) \cdot C(7, 5) = 8!/(6!\cdot 2!) \cdot 7!/(5!\cdot 2!)$$
$$= 28\cdot 21$$
$$= 588$$