Relations

- Relations of Sets
- N-ary Relations
- Relational Databases
- Binary Relation Properties
- Equivalence Relations

Reading (Epp's textbook) 8.1-8.3.

Cartesian Products

- The symbol (a, b) denotes the ordered pair (ordered two-tuple) consisting of a and b together with the specification that
  - a is the first element of the pair
  - b is the second element of the pair.
- $(a, b) = (c, d)$ means that $a = c$ and $b = d$.
- In general two ordered n-tuples $(x_1, x_2, \ldots, x_n)$ and $(y_1, y_2, \ldots, y_n)$ are equal if, and only if, $x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n$.
- **Cartesian product of two sets A and B**:

$$A \times B = \{(a, b) \mid a \in A \text{ and } b \in B\}.$$

- Example: A = {good, bad}, B = {student, prof}

$$A \times B = \{(good, student), (good, prof), (bad, student), (bad, prof)\}.$$

Is $A \times B$ = B $\times A$?

Cartesian Products

Note that:

- $A \times \varnothing = \varnothing$
- $\varnothing \times A = \varnothing$
- For non-empty sets A and B: $A \neq B \Leftrightarrow A \times B \neq B \times A$
- If $|A| = n$ and $|B| = m$, $|A \times B|$ is $n \times m$

➢ **Cartesian product of two or more sets is defined as**:

$$A_1 \times A_2 \times \cdots \times A_n = \{(a_1, a_2 \ldots a_n) \mid a_1 \in A_1, a_2 \in A_2, \ldots, a_n \in A_n\}.$$

Relations

- Let A and B be sets. **A relation $\boldsymbol{R}$ from $\boldsymbol{A}$ to $\boldsymbol{B}$** is a subset of $A \; x \; B$. Given and ordered pair (x, y) in $A \; x \; B$, **$\boldsymbol{x}$ is related to $\boldsymbol{y}$ by $\boldsymbol{R}$**, written $x \; R \; y$, if, and only if, (x, y) is in R.

- The set A is called the **domain** of R and the set B is called its **co-domain**.

- A relation that is a subset of a Cartesian product of **two** sets is called a **binary relation**.

The Inverse of a Relation

Let R be a relation from A to B. Define the **inverse relation** R^{-1} from B to A as follows:

$$R^{-1} = \{(y, x) \in B \times A \mid (x, y) \in R\}.$$

Example:

Let $A = \{2, 4, 6\}$ and $B = \{2, 3\}$. Given any $(x, y) \in A \times B$,

- a relation R from A to B is defined as follows:

  $(x, y) \in R$ means that $\frac{x}{y}$ is an integer.

  - Which ordered pairs are in R?

    R = {(2, 2), (4, 2), (6, 2), (6, 3)}

  - Which ordered pairs are in R^{-1}?

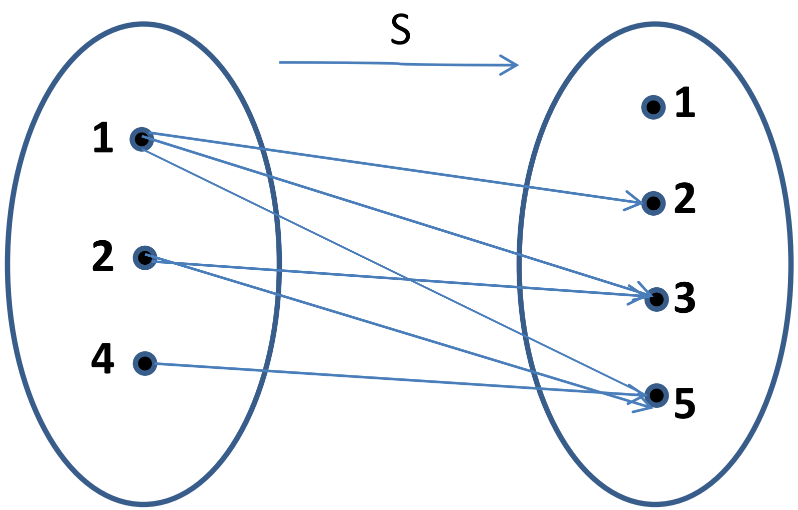
    R = {(2, 2), (2, 4), (2, 6), (3, 6)}

R^{-1} can be described in words as follows: For all *(y, x)* ∈ *B* × *A*, *y* R^{-1} *x* ⇔ *y* is a multiple of *x*.

Arrow Diagram of a Relation

- Let $A = \{1, 2, 4\}$ and $B = \{1, 2, 3, 5\}$ and define relation S from A to B as follows:
  - For all $(x, y) \in A \times B$,

    $(x, y) \in S$ means that x < y



Is relation *S* a function?

A **function** f from a set A to a set B assigns a unique element of B to each element of A.

All functions are relations!
Not every relation is a function!!!

Relation on a Set

- **A relation on a set A** is a relation from A to A.

- When a relation R is defined *on* a set A, the arrow diagram of the relation can be modified so that it becomes a **directed graph (digraph).**

- For all points x and y in A,

  there is an arrow from x to $y \Leftrightarrow x\ R\ y \Leftrightarrow (x, y) \in R$.

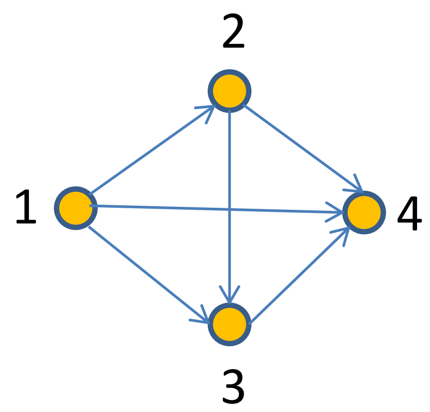
Relation on a Set

Example:

Let A = {1, 2, 3, 4}. Which ordered pairs are in the relation R = {(a, b) | a < b} ?

Solution:

$$\mathrm{R} = \{(1,2),(1,3),(1,4),(2,3),(2,4),(3,4)\}$$



Representing Relations Using Matrices

- If R is a relation from A = $\{a_1, a_2, \ldots, a_m\}$ to B = $\{b_1, b_2, \ldots, b_n\}$, then R can be represented by the zero-one matrix $M_R = [m_{ij}]$ with

$$m_{ij} = 1, \quad \text{if } (a_i, b_j) \in R, \text{ and}$$
$$m_{ij} = 0, \quad \text{if } (a_i, b_j) \notin R.$$

Example: How can we represent the relation *R* from A = {1, 2, 3} to B = { 1, 2} where R = {(2, 1), (3, 1), (3, 2)} as a zero-one matrix?

Solution: The matrix M_R is given by

$$M_R = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 1 \end{bmatrix}$$

Relation on a Set

How many different relations can we define on a set A with *n* elements?

A relation on a set A is a subset of $A \times A$.

How many elements are in $A \times A$?

There are n^2 elements in $A \times A$, so how many subsets (= relations on A) does $A \times A$ have?

The number of subsets that we can form out of a set with *m* elements is 2^m (Power Set).

Therefore, 2^{n^2} subsets can be formed out of $A \times A$.

Answer: We can define 2^{n^2} different relations on A.

N-ary Relations

Definition: Given sets A_1, A_2, …, A_n an **n-ary relation *R*** on these sets is a subset of $A_1 \times A_2 \times \ldots \times An$.

- The sets $A_1, A_2, \ldots, An$ are called the **domains** of the relation, and *n* is called its **degree**.

Example:

Let R = {(a, b, c) | $a = 2b^b$ = 2c with a, b, c ∈ **N**}

What is the degree of R?

✓ The degree of R is 3, so its elements are triples.

What are its domains?

✓ Its domains are all equal to the set of positive integers.

Is (2, 4, 8) in R?

Is (8, 2, 4) in R?

Databases & Relations

- *N*-ary relations form the mathematical foundation for **relational database** theory.
- A database consists of n-tuples called **records**, which are made up of **fields**.
- Relations that represent databases are also called **tables**, since they are often displayed as tables.

Databases & Relations

Example:

Consider a database *S* of students, whose records are represented as 4-tuples with the fields **Student Name**, **ID Number**, **Major**, and **GPA**:

R = {(Ackermann, 231455, CS, 3.88), (Adams, 888323, Physics, 3.45), (Chou, 102147, CS, 3.79), (Rao, 678543, Math, 3.90), (Stevens, 786576, Psych, 3.45)}

Student Name	ID number	Major	GPA
Ackermann	231455	CS	3.88
Adams	888323	Physics	3.45
Chou	102147	CS	3.79
Rao	678543	Math	3.9
Stevens	786576	Psych	3.45

Record

Databases & Relations

A domain of an n-ary relation is called a **primary key** if the n-tuples are uniquely determined by their values from this domain.

- ✓ This means that no two records have the same value from the same primary key.

In our example, which of the fields **Student Name**, **ID Number**, **Major**, and **GPA** are primary keys?

Student Name and **ID Number** are primary keys, because no two records have equal values in these fields.

- ✓ In a real student database, only **ID Number** would be a primary key.

Databases & Relations

- In a database, a primary key should remain **one**, even if new records are added.

- **Combinations of domains** can also uniquely identify n-tuples in an n-ary relation.

- When the values of a **set of domains** determine an n-tuple in a relation, the **Cartesian product** of these domains is called a **composite key**.

- We can apply a variety of **operations** on n-ary relations to form new relations.

Databases & Relations

Example:

In the database language SQL, if the previous student database is denoted S, the result of the query

SELECT Student Name, ID number FROM S WHERE

Major = CS

would be a list of the Student names and ID numbers of all CS's students:

Ackermann, 231455,

Chou, 102147.

This is obtained by taking the intersection of the set $A1 \times A2 \times \{CS\} \times A4$ with the database and then projecting onto the first two coordinates.

Properties of Binary Relations

We will now look at some useful ways to classify relations.

Definition: A relation R on a set A is called **reflexive** if $(a, a) \in R$ for every element $a \in A$.

Example: Are the following relations on {1, 2, 3, 4} reflexive?

- R = {(1, 1), (2, 2), (2, 3), (3, 3), (4, 4)} **YES**
- R = {(1, 1), (2, 2), (3, 3)} **NO**
- R = {(1, 1), (1, 2), (2, 3), (3, 3), (4, 4)} **NO**

Definition: A relation on a set A is called **irreflexive** if $(a, a) \notin R$ for every element $a \in A$.

Properties of Binary Relations

Definition: A relation R on a set A is called **symmetric** if $(b, a) \in R$ whenever $(a, b) \in R$ for all $a, b \in A$.

Example: Are the following relations on {1, 2, 3, 4} symmetric?

- R = {(1, 1), (1, 2), (2, 3), (3, 4), (4, 4)} **NO**
- R = {(1, 2), (2, 2), (3, 1)} **NO**
- R = {(1, 2), (2, 1), (2, 3), (3, 2), (4, 4)} **YES**

Definition: A relation R on a set A is called **asymmetric** if $(a, b) \in R$ implies that $(b, a) \notin R$ for all $a, b \in A$.

Properties of Binary Relations

What do we know about the matrices representing a **relation on a set** (a relation from A to A) ?

✓ They are **square** matrices.

What do we know about matrices representing **reflexive** relations?

✓ All the elements on the **diagonal** of such matrices M_{ref} must be **1s**.

$$M_{ref} = \begin{bmatrix} 1 & & & & & \\ & 1 & & & & \\ & & \cdot & & & \\ & & & \cdot & & \\ & & & & \cdot & \\ & & & & & 1 \end{bmatrix}$$

Properties of Binary Relations

What do we know about the matrices representing **symmetric relations**?

✓ These matrices are symmetric, that is, $M_R = (M_R)^t$.

$$M_R = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

symmetric matrix,
symmetric relation.

$$M_R = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \end{bmatrix}$$

non-symmetric matrix,
non-symmetric relation.

Properties of Binary Relations

Definition: A relation R on a set A is called **transitive** if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

Example: Are the following relations on {1, 2, 3, 4} transitive?

- R = {(1, 1), (1, 2), (2, 3), (3, 4), (4, 4)} **NO**
- R = {(1, 2), (2, 3), (3, 1)} **NO**
- R = {(1, 1), (1, 2), (1, 3), (2, 1), (2, 2), (2, 3), (4, 4)} **YES**

Properties of Binary Relations

Definition: A relation R on a set A is called **antisymmetric** if
a = b whenever $(a, b) \in R$ and $(b, a) \in R$.

Example: Are the following relations on {1, 2, 3} antisymmetric?

- R = {(1, 1), (1, 2), (2, 3)} **YES**
- R = {(1, 3), (1, 2), (3, 1)} **NO**
- R = {(1, 1), (1, 2), (1, 3), (2, 2), (2, 3)} **YES**

Properties of Binary Relations

- ✓ ≤ and = are reflexive, but < is not.
- ✓ = is symmetric, but ≤ is not.
- ✓ ≤ is antisymmetric.
- However, $R = \{(x, y) |\ x + y \leq 3\}$ is not antisymmetric, since (1, 2), (2, 1) ∈ R.
- Note: = is also antisymmetric, i.e., = is symmetric and antisymmetric.
- ✓ < is also antisymmetric, since the precondition of the implication is always false.
- ✓ All three ≤, =, and < are transitive.
- However, $R = \{(x, y) |\ y = 2x\}$ is not transitive.

Properties of Binary Relations

Question: Which of the following relations are reflexive, symmetric, antisymmetric, and/or transitive?

R_1 = {(a,b) | a ≤ b}

R_2 = {(a,b) | a > b}

R_3 = {(a,b) | a = b or a = -b}

R_4 = {(a,b) | a = b}

R_5 = {(a,b) | a = b + 1}

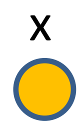
R_6 = {(a,b) | a + b ≤ 3}

	Reflexive	**Symmetric**	**Antisymmetric**	**Transitive**
$\boldsymbol{R_1}$	Yes	No	Yes	Yes
$\boldsymbol{R_2}$	No	No	Yes	Yes
$\boldsymbol{R_3}$	Yes	Yes	No	Yes
$\boldsymbol{R_4}$	Yes	Yes	Yes	Yes
$\boldsymbol{R_5}$	No	No	Yes	No
$\boldsymbol{R_6}$	No	Yes	No	No

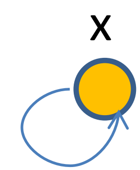
Properties of Relations Using Graphs

What do we could derive about the graphs representing a **relation on a set** (a relation from A to A)?

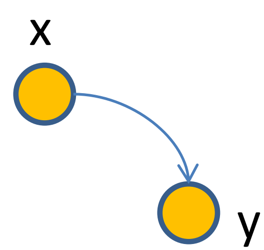
A relation is reflexive if for each point x ...



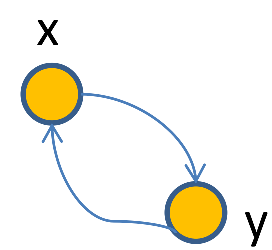
... there is a loop at x:



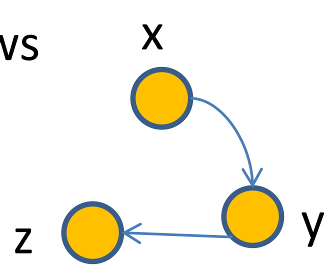
A relation is symmetric if whenever there is an arrow from x to y ...



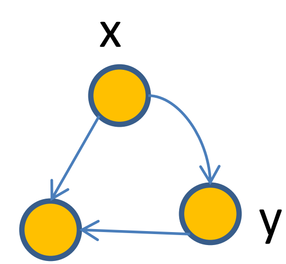
... there is also an arrow from y back to x:



A relation is transitive if whenever there are arrows from x to y and y to z ...



...there is also an arrow from x to z:



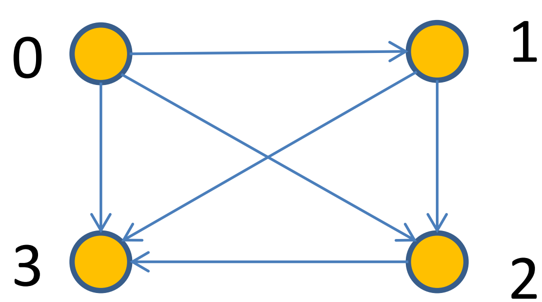
Transitive Closure of a Relation

Let A be a set and R a relation on A. The **transitive closure** of R is the relation R^t on A that satisfies the following three properties:

1. R^t is transitive.
2. $R \subseteq R^t$.
3. If S is any other transitive relation that contains R, then $R^t \subseteq S$.

Example:

Let A = {0, 1, 2, 3} and consider the relation R defined on A as R = {(0, 1), (1, 2), (2, 3)}. Find the transitive closure of R.



R^t ={(0, 1), (0, 2), (0, 3), (1, 2), (1, 3), (2, 3)}.

Combining Relations

- Relations are sets, and therefore, we can apply the usual **set operations** to them.

- If we have two relations R_1 and R_2, and both of them are from a set A to a set B, then we can combine them to $R_1 \cup R_2$, $R_1 \cap R_2$, or $R_1 - R_2$.

- In each case, the result will be **another relation from A to B**.

Combining Relations

Example:

Let the relations R and S be represented by the matrices

$$M_R = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \qquad M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

What are the matrices representing R∪S and R∩S?

Solution: These matrices are given by

$$M_{R\cup S} = M_R \vee M_S = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix} \qquad M_{R\cap S} = M_R \wedge M_S = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Combining Relations

Let R_1 and R_2 be transitive relations on a set A. Does it follow that $R_1 \cup R2$ is transitive?

Solution:

No. Here is a counterexample:

$$A = \{1, 2\}, \qquad R_1 = \{(1, 2)\}, \qquad R_2 = \{(2, 1)\}$$

- Therefore, $R_1 \cup R_2 = \{(1, 2), (2, 1)\}$
- Notice that R_1 and R_2 are both transitive (vacuously, since there are no two elements satisfying the conditions of the property). However $R_1 \cup R_2$ is not transitive.
- If it were it would have to have (1, 1) and (2, 2) in $R_1 \cup R_2$.

Combining Relations

… and there is another important way to combine relations.

- **Definition:** Let R be a relation from a set A to a set B and S a relation from B to a set C. The **composite** of R and S is the relation consisting of ordered pairs (a, c), where $a \in A$, $c \in C$, and for which there exists an element $b \in B$ such that $(a, b) \in R$ and $(b, c) \in S$. We denote the composite of R and S by $\mathbf{S \circ R}$.

- In other words, if relation R contains a pair (a, b) and relation S contains a pair (b, c), then $S \circ R$ contains a pair (a, c).

Combining Relations

Example:

Let D and S be relations on A = {1, 2, 3, 4}.

$D = \{(a, b) \mid b = 5 - a\}$ "b equals (5 – a)"

$S = \{(a, b) \mid a < b\}$ "a is smaller than b"

D = {(1, 4), (2, 3), (3, 2), (4, 1)}

S = {(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4)}

S∘D = { (2, 4), (3, 3), (3, 4), (4, 2), (4, 3), (4, 4)}

✓ D maps an element a to the element (5 – a), and afterwards S maps (5 – a) to all elements larger than (5 – a), resulting in **$S \circ D = \{(a,b) \mid b > 5 - a\}$** or **$S \circ D = \{(a,b) \mid a + b > 5\}$.**

Powers of a Relation

Definition: Let R be a relation on the set A. The powers R^n, n = 1, 2, 3, …, are defined inductively by

- $R^1 = R$
- $R^{n+1} = R^n \circ R$

In other words:

- $R^n = R \circ R \circ \ldots \circ R$ (n times the letter R)

Powers of a Relation

Theorem: The relation R on a set A is transitive if and only if $R^n \subseteq R$ for all positive integers n.

Remember the definition of transitivity:

Definition: A relation R on a set A is called transitive if whenever $(a, b) \in R$ and $(b, c) \in R$, then $(a, c) \in R$ for $a, b, c \in A$.

- The composite of R with itself contains exactly these pairs (a, c).
- ✓ Therefore, for a transitive relation R, $R \circ R$ does not contain any pairs that are not in R, so $R \circ R \subseteq R$.
- ✓ Since $R \circ R$ does not introduce any pairs that are not already in R, it must also be true that $(R \circ R) \circ R \subseteq R$, and so on, so that $R^n \subseteq R$.

Equivalence Relation

❖ **Equivalence relations** are used to relate objects that are similar in some way.

Definition: A relation on a set A is called an **equivalence relation** if it is reflexive, symmetric, and transitive.

✓ Two elements that are related by an equivalence relation R are called **equivalent**.

Equivalence Relation

- Since R is **symmetric**, a is equivalent to b whenever b is equivalent to a.

- Since R is **reflexive**, every element is equivalent to itself.

- Since R is **transitive**, if a and b are equivalent and b and c are equivalent, then a and c are equivalent.

Obviously, these three properties are necessary for a reasonable definition of equivalence.

Equivalence Relation

Example:

Let $R = \{(a, b) \in Z^{+} \times Z^{+} | \, a \; divides \; b\}$.

Solution:

To be an equivalence relation, *R* should be reflexive, transitive, and symmetric.

Is it reflexive?

YES

Is it transitive?

YES

Is it symmetric?

NO

Equivalence Relation

Example: Suppose that R is the relation on the set of strings that consist of English letters such that aRb if and only if l(a) = l(b), where l(x) is the length of the string x. Is R an equivalence relation?

Solution:

- ✓ R is reflexive, because l(a) = l(a) and therefore aRa for any string a.
- ✓ R is symmetric, because if l(a) = l(b) then l(b) = l(a), so if aRb then bRa.
- ✓ R is transitive, because if l(a) = l(b) and l(b) = l(c), then l(a) = l(c), so aRb and bRc implies aRc.
- ✓ R is an equivalence relation.

Equivalence Classes

Definition: Let R be an equivalence relation on a set A. The set of all elements that are related to an element a of A is called the **equivalence class** of a.

The equivalence class of a with respect to R is denoted by $[\boldsymbol{a}]_{\boldsymbol{R}}$.

When only one relation is under consideration, we will delete the subscript R and write $[\boldsymbol{a}]$ for this equivalence class.

If $b \in [a]_R$, b is called a **representative** of this equivalence class.

Equivalence Classes

Example:

In the previous example (strings of identical length), what is the equivalence class of the word mouse, denoted by [mouse]?

Solution: [mouse] is the set of all English words containing five letters.

For example, 'horse' would be a representative of this equivalence class.

Equivalence Classes

Theorem: Let R be an equivalence relation on a set A. The following statements are equivalent:

aRb

[a] = [b]

$[a] \cap [b] \neq \emptyset$

Definition: A **partition** of a set S is a collection of disjoint nonempty subsets of S that have S as their union. In other words, the collection of subsets A_i, $i \in I$, forms a partition of S if and only if

(i) $A_i \neq \emptyset$ for $i \in I$

(ii) $A_i \cap A_j = \emptyset$, if $i \neq j$

(iii) $\cup_{i \in I} A_i = S$

Equivalence Classes

Theorem: Let R be an equivalence relation on a set S. Then the **distinct equivalence classes** of R form a **partition** of S. Conversely, given a partition $\{A_i \mid i \in I\}$ of the set S, there is an equivalence relation R that has the sets A_i, $i \in I$, as its equivalence classes.

Example: Let us assume that Frank, Suzanne and George live in Boston, Stephanie and Max live in Lübeck, and Jennifer lives in Sydney.

Let R be the **equivalence relation** {(a, b) | a and b live in the same city} on the set P = {Frank, Suzanne, George, Stephanie, Max, Jennifer}.

Then R = {(Frank, Frank), (Frank, Suzanne),(Frank, George), (Suzanne, Frank), (Suzanne, Suzanne), (Suzanne, George), (George, Frank), (George, Suzanne), (George, George), (Stephanie, Stephanie), (Stephanie, Max), (Max, Stephanie), (Max, Max), (Jennifer, Jennifer)}.

Equivalence Classes

- Then the ***distinct* equivalence classes** of R are:

{{Frank, Suzanne, George}, {Stephanie, Max}, {Jennifer}}.

- And this is a **partition** of P.

✓ The distinct equivalence classes of any equivalence relation R defined on a set S constitute a partition of S, because every element in S is assigned to **exactly one** of the equivalence classes.

Congruence modulo m

We say that ***a* is congruent to *b* modulo *m*** and write $a \equiv b(\,mod\ m)$ iff m |(a-b).

Let m > 1 be an integer. Show that the relation $R = \{(a, b) \mid a \equiv b(\,mod\ m)\}$ is an equivalence on the set of integers.

Proof:

- ✓ *Reflexivity:* $a \equiv a(\,mod\ m)$ since $a - a$ = 0 is divisible by m.
- ✓ *Symmetry:* Suppose $(a, b) \in R$. Then m divides $a - b$. Thus there exists some integer *k* s.t. $a - b = km$. Therefore, $b - a = (-k)m$. So m divides $b - a$ and thus $b \equiv a(mod\ m)$, and finally $(b, a) \in R$.
- ✓ *Transitivity:* If $(a, b) \in R$ and $(b, c) \in R$ then $a \equiv b(\,mod\ m)$ and $b \equiv c(mod\ m)$. So m divides both $a - b$ and $b - c$. Hence there exist integers *k, r* with $a - b = km$ and $b - c = rm$. By adding these two equations we obtain

$$a - c = (a - b) + (b - c) = km + rm = (k + r)m.$$

Therefore, $a \equiv c(\,mod\ m)$ and $(a, c) \in R$.

Congruence modulo m

Example:

Let R be the relation $\{(a, b) \mid a \equiv b \pmod 3\}$ on the set of integers.

Is R an equivalence relation?

Yes, R is reflexive, symmetric, and transitive.

What are the distinct equivalence classes of R ?

$\{\{\ldots, -6, -3, 0, 3, 6, \ldots\},$
$\{\ldots, -5, -2, 1, 4, 7, \ldots\},$
$\{\ldots, -4, -1, 2, 5, 8, \ldots\}\}$