Boolean Algebra and Functions

- Boolean Algebra

Reading (Epp's textbook)
6.4

- Functions

Reading (Epp's textbook)
7.1 - 7.2

Boolean Algebra

A **Boolean algebra** is a set B together with two operations, generally denoted + and ·, such that for all a and b in B both $a + b$ and $a \cdot b$ are in B and the following properties hold:

1. *Commutative Laws:* For all a and b in B,
(i) $a + b = b + a$ and (ii) $a \cdot b = b \cdot a$.
2. *Associative Laws:* For all a, b, and c in B,
(i) $(a + b) + c = a + (b + c)$ and (ii) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.
3. *Distributive Laws:* For all a, b, and c in B,
(i) $a + (b \cdot c) = (a + b) \cdot (a + c)$ and (ii) $a \cdot (b + c) = (a \cdot b) + (a \cdot c)$.

Boolean Algebra

4. *Identity Laws:* There exist distinct elements 0 and 1 in *B* such that for all a in *B*,

(i) $a + 0 = a$ and (b) $a \cdot 1 = a$.

5. *Complement Laws:* For each a in *B*, there exists an element in *B*, denoted $\bar{a}$ and called the **complement** or **negation** of a, such that

(i) $a + \bar{a} = 1$ and (ii) $a \cdot \bar{a} = 0$.

Properties of a Boolean Algebra

Let B be any Boolean algebra.

1. *Uniqueness of the Complement Law:* For all a and x in B, if $a + x = 1$ and $a \cdot x = 0$ then $x = \bar{a}$.

2. *Uniqueness of* 0 *and* 1: If there exists x in B such that $a + x = a$ for all a in B, then $x = 0$, and if there exists y in B such that $a \cdot y = a$ for all a in B, then $y = 1$.

3. *Double Complement Law:* For all $a \in B$, $\overline{(\bar{a})} = a$.

4. *Idempotent Law:* For all $a \in B$,

(i) $a + a = a$ and (ii) $a \cdot a = a$.

5. *Universal Bound Law:* For all $a \in B$,

(i) $a + 1 = 1$ and (ii) $a \cdot 0 = 0$.

Properties of a Boolean Algebra

6. *De Morgan's Laws:* For all *a* and $b \in B$,

(i) $\overline{a+b} = \bar{a} \cdot \bar{b}$ and (ii) $\overline{a \cdot b} = \bar{a} + \bar{b}$

7. *Absorption Laws:* For all *a* and $b \in B$,

(i) $(a + b) \cdot a = a$ and (ii) $(a \cdot b) + a = a$.

8. *Complements of* 0 *and* 1:

(i) $\overline{0} = 1$ and (ii) $\overline{1} = 0$.

Boolean Functions

Definition: Let B = {0, 1}. The variable x is called a **Boolean variable** if it assumes values only from B.

A function from B^n, the set $\{(x_1, x_2, \ldots, x_n) \mid x_i \in B, 1 \leq i \leq n\}$, to B is called a **Boolean function of degree n**.

$(x_1, x_2, \ldots, x_n)$ is an ordered n-tuple!

- ✓ Boolean functions can be represented using expressions made up from the Boolean variables and Boolean operations.

Boolean Expressions

The **Boolean expressions** in the variables $x_1, x_2, \ldots, x_n$ are defined recursively as follows:

- $0, 1, x_1, x_2, \ldots, x_n$ are Boolean expressions.
- If E_1 and E_2 are Boolean expressions, then $(\overline{E_1})$, $(\overline{E_2})$, $(E_1 \cdot E_2)$, and $(E_1 + E_2)$ are Boolean expressions.

➢ We can create Boolean expression in the variables x, y, and z using the "building blocks"
0, 1, x, y, and z, and the construction rules:

Since x and y are Boolean expressions, so is $x \cdot y$.

Since z is a Boolean expression, so is $\bar{z}$.

Since $x \cdot y$ and $\bar{z}$ are expressions, so is $x \cdot y + \bar{z}$.

… and so on…

Boolean Functions & Expressions

Example:

Give a Boolean expression for the Boolean function F(x, y) as defined by the following Input-Output table:

x	**y**	**F(x , y)**
1	1	0
1	0	1
0	1	0
0	0	0

Possible solution: $F(x, y) = x \cdot \bar{y}$

Functions defined on General Sets

- **A function $\boldsymbol{F}$ from a set $\boldsymbol{A}$ to a set $\boldsymbol{B}$**, denoted $f: A \rightarrow B$, is a relation with **domain** A and **co-domain** B that satisfies the following two properties:
  1. For every element x in A, there is an element y in B such that $(x, y) \in F$.
  2. For all elements x in A and y and z in B,

     If $(x, y) \in F$ and $(x, z) \in F$, then y = z.

(note: Here, "→" has nothing to do with if… then)

- If $f(x) = y$, we say that *y* is the **image** of *x* and *x* is the **pre-image** of *y*.
- **range of $\boldsymbol{f}$** $= \boldsymbol{image\ of\ x} = \{y \in B \mid y = f(x), for\ some\ x\ in\ A\}$.
- **the inverse image of $\boldsymbol{y}$** $= \boldsymbol{preimage\ of\ y} = \{x \in A \mid f(x) = y\}$.

Functions

Let us take a look at the function $f: P \rightarrow C$ with

- P = {Linda, Max, Kathy, Peter}
- C = {Boston, New York, Hong Kong, Moscow}

f(Linda) = Moscow

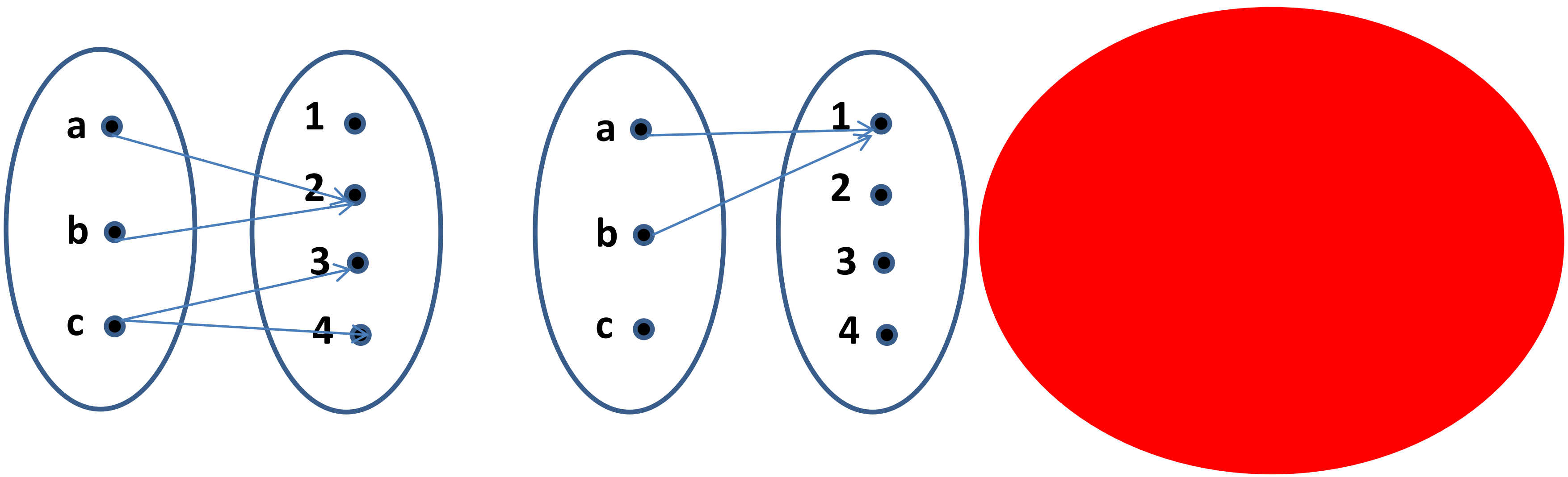
f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = New York

Here, the range of *f* is *C*.

Functions Arrow Diagrams



- Which of the arrow diagrams define functions from X = {a, b, c} to Y = {1, 2, 3, 4}?

Function's formula

If the domain of our function f is large, it is convenient to specify f with a **formula**, e.g.:

f:$\mathbf{R} \rightarrow \mathbf{R}$

f(x) = 2x

This leads to:

f(1) = 2

f(3) = 6

f(-3) = -6

…

Equality of Functions

- If $F: X \to Y$ and $G: X \to Y$ are functions, then $F = G$ if, and only if, $F(x) = G(x)$ for all $x \in X$.
- Example: Let $J_3 = \{0, 2, 4\}$, $J_4 = \{0, 1\}$, and define functions f and g from J_3 to J_4 , as follows: For all x in J_3,

  $f(x) = (x^2 + x + 1)\ mod\ 3$ and $g(x) = (x + 2)^2\ mod\ 3$

  Does $f = g$?

x	$x^2 + x + 1$	$(x^2 + x + 1)\ mod\ 3$	$(x + 2)^2$	$(x + 2)^2\ mod\ 3$
0	1	$1\ mod\ 3 = 1$	4	$4\ mod\ 3 = 1$
2	7	$7\ mod\ 3 = 1$	16	$16\ mod\ 3 = 1$
4	21	$21\ mod\ 3 = 0$	36	$36\ mod\ 3 = 0$

Yes, the table of values shows that $f(x) = g(x)$ for all $x \in J_3$.

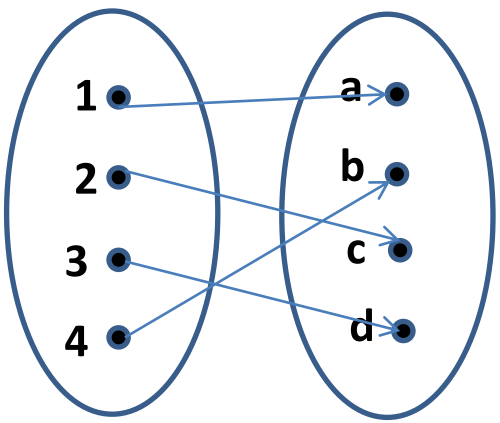
Functions Acting on Sets

If $f : X \rightarrow Y$ is a function and $A \subseteq X$ and $C \subseteq Y$, then

$$f(A) = \{y \in Y \mid y = f(x) \text{ for some } x \text{ in } A\}$$

$$\text{and } f^{-1}(C) = \{x \in X \mid f(x) \in C\}.$$

$f(A)$ is called the **image of *A***, and $f^{-1}(C)$ is called the **inverse image of *C***.



Let A = {1, 3}, C = {a, b, c}

Find F(A) and $F^{-1}(C)$

F(A) = {a, d}

$F^{-1}(C)$ = { 1, 4, 2}

Function's Properties

A function $f: A \rightarrow B$ is said to be **one-to-one** (or **injective**), if and only if

$$\forall x, y \in A \ (f(x) = f(y) \rightarrow x = y)$$

In other words: f is one-to-one if and only if it does not map two distinct elements of A onto the same element of B.

Function's Properties

And again…

f(Linda) = Moscow

f(Max) = Boston

f(Kathy) = Hong Kong

f(Peter) = Boston

Is f one-to-one?

No, Max and Peter are mapped onto the same element of the image.

What about …

g(Linda) = Moscow

g(Max) = Boston

g(Kathy) = Hong Kong

g(Peter) = New York

Is g one-to-one?

Yes, each element is assigned a unique element of the image.

Function's Properties

How can we prove that a function f is one-to-one?

Whenever you want to prove something, first take a look at the relevant definition(s):

$$\forall x, y \in A\ (f(x) = f(y) \rightarrow x = y)$$

Example:

$$f: \mathbf{R} \rightarrow \mathbf{R}$$
$$f(x) = x^2$$

Disproof by counterexample:

f(2) = f(-2), but 2 $\neq$ -2, so f is not one-to-one.

What about:

$$f: \mathbf{R} \rightarrow \mathbf{R}$$
$$f(x) = 5x - 3$$

Proving one-to-one example

Consider the function $f\colon Z \rightarrow Z$ defined as:

$$f(x) = \begin{cases} 3x + 1, & if\ x \geq 0 \\ -3x + 2, & if\ x < 0 \end{cases}$$

Prove that f is injective.

We need to show that if $x \neq y$, then $f(x) \neq f(y)$.

What proof technique do we need to use?

Function's Properties

A function f:A→B with A,B $\subseteq$ R is called:

- **strictly increasing**, if

$$\forall x, y \in A\ (x < y \rightarrow f(x) < f(y)),$$

and

- **strictly decreasing**, if

$$\forall x, y \in A\ (x < y \rightarrow f(x) > f(y)).$$

Obviously, a function that is either strictly increasing or strictly decreasing is one-to-one.

Function's Properties

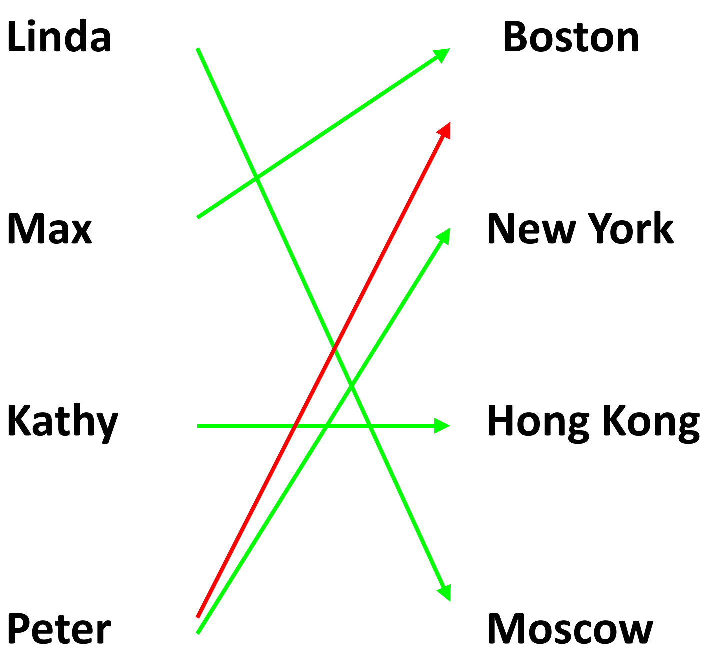
A function $f: A \rightarrow B$ is called **onto**, or **surjective**, if and only if for every element $b \in B$ there is an element $a \in A$ with $f(a) = b$.

In other words, *f* is onto if and only if its range is its entire codomain.

A function $f:\ A \rightarrow B$ is a **one-to-one correspondence**, or a **bijection**, if and only if it is both one-to-one and onto.

Obviously, if *f* is a bijection and *A* and *B* are finite sets, then |A| = |B|.

Function’s Properties



- Is f injective?
- No.
- Yes.
- Is f surjective?
- No.
- Yes.
- Is f bijective?
- No.
- Yes.

Inverse Function

- An interesting property of bijections is that they have an **inverse function**.

- The **inverse function** of the bijection $f: A \rightarrow B$ is the function $f^{-1}: B \rightarrow A$ with

$$f^{-1}(b) = a \text{ whenever } f(a) = \text{b.}$$

Inverse Function

Example:

f(Linda) = Moscow
f(Max) = Boston
f(Kathy) = Hong Kong
f(Peter) = Lübeck
f(Helena) = New York

Assume, f is bijective.

Then the inverse function f^{-1} is given by:

f^{-1}(Moscow) = Linda
f^{-1}(Boston) = Max
f^{-1}(Hong Kong) = Kathy
f^{-1}(Lübeck) = Peter
f^{-1}(New York) = Helena

Inversion is only possible for bijections (= invertible functions)