Recurrence Relations

- Recurrence Relations

Reading (Epp's textbook)
5.6 – 5.8

Recurrence Relations

- A **recurrence relation** for a sequence $a_0, a_1, a_2, \ldots$ ($\{a_n\}$) is a formula that relates each term a_k to certain of its predecessors $a_{k-1}, a_{k-2}, \ldots, a_{k-i}$ (one or more of the previous terms of the sequence), where i is an integer with $k - i \geq 0$.
- In other words, a recurrence relation is like a recursively defined sequence, but **without specifying any initial values (initial conditions)**.
- Therefore, the same recurrence relation can have (and usually has) **multiple solutions**.
- If **both** the initial conditions and the recurrence relation are specified, then the sequence is **uniquely** determined.

Recurrence Relations

Example:

Consider the recurrence relation

$a_n = 2a_{n-1} – a_{n-2}$ for $n = 2, 3, 4, \ldots$

Is the sequence $\{a_n\}$ with $a_n= 3n$ a solution of this recurrence relation?

For $n \geq 2$ we see that

$2a_{n-1} – a_{n-2} = 2(3(n – 1)) – 3(n – 2) = 3n = a_n$.

Therefore, $\{a_n\}$ with $a_n= 3n$ is a solution of the recurrence relation.

Recurrence Relations

Is the sequence $\{a_n\}$ with $a_n = 5$ a solution of the same recurrence relation?

For $n \geq 2$ we see that
$2a_{n-1} - a_{n-2} = 2 \cdot 5 - 5 = 5 = a_n$.

Therefore, $\{a_n\}$ with $a_n=5$ is also a solution of the recurrence relation.

Recurrence Relations

Fibonacci Numbers:

Consider the recurrence relation

$$F_k = F_{k-1} + F_{k-2}$$

for all integers $k \geq 2$

Initial conditions $F_0 = 1, F_1 = 1$.

Compute F_2, F_3 and so forth through F_9.

The method of **Iteration**.

- $F_2 = F_1 + F_0 = 1 + 1 = 2$
- $F_3 = F_2 + F_1 = 2 + 1 = 3$
- $F_4 = F_3 + F_2 = 3 + 2 = 5$
- $F_5 = F_4 + F_3 = 5 + 3 = 8$
- $F_6 = F_5 + F_4 = 8 + 5 = 13$
- $F_7 = F_6 + F_5 = 13 + 8 = 21$
- $F_8 = F_7 + F_6 = 21 + 13 = 34$
- $F_9 = F_8 + F_7 = 21 + 34 = 55$

Modeling with Recurrence Relations

Example:

Someone deposits \$10,000 in a savings account at a bank yielding 5% per year with interest compounded annually. How much money will be in the account after 30 years?

Solution:

Let P_n denote the amount in the account after n years.

How can we determine P_n on the basis of P_{n-1}?

We can derive the following **recurrence relation**:

$$P_n = P_{n-1} + 0.05P_{n-1} = 1.05P_{n-1}.$$

Modeling with Recurrence Relations

The initial condition is $P_0 = 10{,}000$.

Then we have:

$P_1 = 1.05P_0$

$P_2 = 1.05P_1 = (1.05)^2P_0$

$P_3 = 1.05P_2 = (1.05)^3P_0$

…

$P_n = 1.05P_{n-1} = (1.05)^nP_0$

- We now have an **explicit formula** to calculate P_n for any natural number *n* and can avoid the **iteration**.
- Let us use this formula to find P_{30} under the initial condition $P_0 = 10{,}000$:

  $P_{30} = (1.05)^{30} \cdot 10{,}000 = 43{,}219.42$

Explicit Formulas

- A sequence *a*0*, a*1*, a*2*, . . .* is called an **arithmetic sequence** if, and only if, there is a constant *d* such that

  $a_n = a_0 + d \times n$, for all integers *n* ≥ 0.

- A sequence *a*0*, a*1*, a*2*, . . .* is called a **geometric sequence** if, and only if, there is a constant *r* such that

  $a_n = a_0 \times r^n$, for all integers *n* ≥ 0.

Correctness of an Explicit Formula

- **Check the Correctness of an explicit formula by a Mathematical Induction!!**

Example:

Let $c_0, c_1, c_2, \ldots$ be the sequence defined as follows:

$$c_k = 2c_{k-1} + k \quad \text{for all integers } k \geq 1,$$
$$c_0 = 1.$$

Suppose your calculations suggest that $c_0, c_1, c_2, \ldots$ satisfies the following explicit formula:

$$c_n = 2^n + n \text{ for all integers } n \geq 0.$$

Is this formula correct?

Solving Recurrence Relations

- ✓ In general, we would prefer to have an **explicit formula** to compute the value of a_n rather than conducting *n* iterations.

For one class of recurrence relations, we can obtain such formulas in a systematic way.

Those are the recurrence relations that express the terms of a sequence as **linear combinations** of previous terms.

Solving Recurrence Relations

Definition: A linear **homogeneous** recurrence relation of degree *k* with constant coefficients is a recurrence relation of the form:

$$a_n = c_1a_{n-1} + c_2a_{n-2} + \ldots + c_ka_{n-k},$$

Where $c_1, c_2, \ldots, c_k$ are real numbers, and $c_k \neq 0$.

Solving Recurrence Relations

Examples:

- The recurrence relation $P_n = (1.05)P_{n-1}$ is a linear homogeneous recurrence relation of **degree one**.
- The recurrence relation $f_n = f_{n-1} + f_{n-2}$ is a linear homogeneous recurrence relation of **degree two**.
- The recurrence relation $a_n = a_{n-5}$ is a linear homogeneous recurrence relation of **degree five**.

Solving Recurrence Relations

Examples

Which of these are linear homogenous recurrence relations with constant coefficients?

- $a_n = a_{n-1} + 2a_{n-5}$ Yes; with coefficients 1 and 2 (degree 5)
- $a_n = 2a_{n-2} + 5$ No; not homogenous
- $a_n = a_{n-1} + n$ No; not homogenous
- $a_n = a_{n-1} \times a_{n-2}$ No; not linear
- $a_n = n \times a_{n-1}$ No; no constant coefficients.

Solving Recurrence Relations

- Cook-book recipe for solving linear homogenous recurrence relations with constant coefficients.
- We try to find solutions of the form **$a_n = r^n$**, where r is a constant.

$a_n = r^n$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} + \dots + c_k a_{n-k} \text{ if and only if}$$

$$r^n = c_1 r^{n-1} + c_2 r^{n-2} + \dots + c_k r^{n-k}.$$

- Divide this equation by r^{n-k} and subtract the right-hand side from the left:

$$r^k - c_1 r^{k-1} - c_2 r^{k-2} - \dots - c_{k-1} r - c_k = 0$$

This is called the **characteristic equation** of the recurrence relation.

Solving Recurrence Relations

- The solutions of this equation are called the **characteristic roots** of the recurrence relation.

- Let us now consider linear homogeneous recurrence relations of **degree two**.

Theorem: Let c_1 and c_2 be real numbers. Suppose that $r^2 – c_1r – c_2 = 0$ has two distinct roots r_1 and r_2.

Then the sequence $\{a_n\}$ is a solution of the recurrence relation

$a_n = c_1a_{n-1} + c_2a_{n-2}$ if and only if $a_n = C \times r_1^n + D \times r_2^n$ for $n = 0, 1, 2, \ldots$, where C and D are constants.

See pp. 321 and 322 for the proof.

Solving Recurrence Relations

Example:

What is the solution of the recurrence relation $a_n = a_{n-1} + 2a_{n-2}$ with $a_0 = 2$ and $a_1 = 7$?

Solution:

The characteristic equation of the recurrence relation is $r^2 - r - 2 = 0$.

Its roots are $r = 2$ and $r = -1$.

Hence, the sequence $\{a_n\}$ is a solution to the recurrence relation if and only if:

$a_n = C\,2^n + D\,(-1)^n$ for some constants C and D.

Solving Recurrence Relations

Given the equation $a_n = C\,2^n + D\,(-1)^n$ and the initial conditions $a_0 = 2$ and $a_1 = 7$, it follows that

$a_0 = 2 \Rightarrow C + D = 2$

$a_1 = 7 \Rightarrow C\cdot 2 + D\cdot(-1) = 7$

Solving these two equations gives us

$C = 3$ and $D = -1$.

Therefore, the solution to the recurrence relation and initial conditions is the sequence $\{a_n\}$ with

$a_n = 3\cdot 2^n - (-1)^n$.

Solving Recurrence Relations

Example:

Give an explicit formula for the Fibonacci numbers.

Solution:

The Fibonacci numbers satisfy the recurrence relation $f_n = f_{n-1} + f_{n-2}$ with initial conditions $f_0 = 1$ and $f_1 = 1$.

The characteristic equation of the recurrence relation is $r^2 - r - 1 = 0$.

Its roots are:

$$r_1 = \frac{1+\sqrt{5}}{2}, \quad r_2 = \frac{1-\sqrt{5}}{2}$$

Solving Recurrence Relations

Therefore, the Fibonacci numbers are given by

$$f_n = C\left(\frac{1+\sqrt{5}}{2}\right)^n + D\left(\frac{1-\sqrt{5}}{2}\right)^n$$

$f_0 = 0 \Rightarrow C + D = 1$

$f_1 = 1 \Rightarrow \quad f_1 = C\left(\frac{1+\sqrt{5}}{2}\right) + D\left(\frac{1-\sqrt{5}}{2}\right) = 1$

The unique solution to this system of two equations and two variables is:

$$C = \frac{1+\sqrt{5}}{2\sqrt{5}}, \quad D = -\frac{1-\sqrt{5}}{2\sqrt{5}}$$

Solving Recurrence Relations

So finally we obtained an explicit formula for the Fibonacci numbers:

$$f_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}$$

Solving Recurrence Relations

But what happens if the characteristic equation has only one root?

How can we then match our equation with the initial conditions a_0 and a_1 ?

Theorem: Let c_1 and c_2 be real numbers with $c_2 \neq 0$. Suppose that $r^2 - c_1 r - c_2 = 0$ has only one root r_0.
A sequence $\{a_n\}$ is a solution of the recurrence relation

$$a_n = c_1 a_{n-1} + c_2 a_{n-2} \text{ if and only if}$$

$a_n = C \times r_0^n + D \times n \times r_0^n$, for $n = 0, 1, 2, \ldots$, where C and D are constants.

Solving Recurrence Relations

Example:

What is the solution of the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with $a_0 = 1$ and $a_1 = 6$?

Solution:

The only root of $r^2 - 6r + 9 = 0$ is $r_0 = 3$.

Hence, the solution to the recurrence relation is

$a_n = C\,3^n + D\,n3^n$ for some constants α_1 and α_2.

To match the initial condition, we need

$a_0 = 1 \Rightarrow C = 1$

$a_1 = 6 \Rightarrow C\cdot 3 + D\cdot 3 = 6$

Solving these equations yields $C = 1$ and $D = 1$.

Consequently, the overall solution is given by

$$a_n = 3^n + n3^n.$$