Mathematical Induction (Cont.)

- Sequences

Reading (Epp's textbook)
5.1

- Mathematical Induction Examples

Reading (Epp's textbook)
5.2 – 5.3

- Strong Mathematical Induction

Reading (Epp's textbook)
5.4

Sequences

A **sequence** is a function whose domain is either all the integers between two given integers or all the integers greater than or equal to a given integer.

$$a_m, a_{m+1}, a_{m+2}, \dots, a_n$$

- a_k is called a **term**
- *k* is called **index**
- *m* is the index of the **initial term**
- *n* is the index of the **final term**

Infinite sequence: $a_m, a_{m+1}, a_{m+2}, \dots$

Sequences

$$a_0, a_1, a_2, a_3, a_4, a_5 \ldots$$

1, 2, 4, 8, 16, 32

It is convenient to describe a sequence with a **formula**.

Any pattern?

$$a_k = 2^k$$

What are the formulas that describe the following sequences $a_1, a_2, a_3, \ldots$?

1, 3, 5, 7, 9, …	$a_n = 2n - 1$
-1, 1, -1, 1, -1, …	$a_n = (-1)^n$
2, 5, 10, 17, 26, …	$a_n = n^2 + 1$
3, 9, 27, 81, 243, …	$a_n = 3^n$

Summations

What does $\sum_{j=m}^{n} a_j$ stand for?

❖ It represents the sum

$$a_m + a_{m+1} + a_{m+2} + \cdots + a_n$$

The variable *j* is called the **index of summation**, running from its **lower limit** *m* to its **upper limit** *n*. We could as well have used any other letter to denote this index.

Summations

How can we express the sum of the first 1000 terms of the sequence with formula $a_n=n^2$ for n = 1, 2, 3, … ?

- We write it as $\sum_{j=1}^{1000} j^2$

What is the value of $\sum_{j=1}^{5} j^2$?

- It is 1 + 4 + 9 + 16 + 25 = 55

Products

What does $\prod_{j=m}^{n} a_j$ stand for?

- It represents the product

$$a_m \times a_{m+1} \times a_{m+2} \times \cdots \times a_n$$

How can we express the product of the first 5 terms of the sequence with formula a_n= n for n = 1, 2, 3, 4, 5 ?

- We write it as $\prod_{j=1}^{5} j$

What is the value of $\prod_{j=1}^{5} j$?

- It is 1 x 2 x 3 x 4 x 5 = 120

Properties of Summations and Products

If $a_m, a_{m+1}, a_{m+2}, \ldots$ and $b_m, b_{m+1}, b_{m+2}, \ldots$ are sequences of real numbers and *c* is any real number, then the following equations hold for any integer *n* ≥ *m*:

1. $\sum_{k=m}^{n} a_k + \sum_{k=m}^{n} b_k = \sum_{k=m}^{n}(a_k + b_k)$

2. $c \times \sum_{k=m}^{n} a_k = \sum_{k=m}^{n} c \times a_k$

3. $(\prod_{k=m}^{n} a_k) \times (\prod_{k=m}^{n} b_k) = \prod_{k=m}^{n}(a_k \times b_k)$

Factorial

For each positive integer *n*, the quantity ***n* factorial** denoted ***n*!,** is defined to be the product of all the integers from 1 to *n*:

$$n! = n \cdot (n-1) \cdots 3 \cdot 2 \cdot 1.$$

- **Zero factorial,** denoted 0!, is defined to be 1:

$$0! = 1.$$

Example

$6! = 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$

$$\frac{5!}{2! \times 3!} = \frac{5 \times 4 \times 3!}{2! \times 3!} = \frac{5 \times 4}{2 \times 1} = 10$$

“n choose r”

Let *n* and *r* be integers with $0 \le r \le n$. The symbol

$$\binom{n}{r}$$

is read **“*n* choose *r*”** and represents the number of subsets of size *r* that can be chosen from a set with *n* elements.

- Formula for computing $\binom{n}{r}$

$$\binom{n}{r} = \frac{n!}{r! \times (n-r)!}$$

More Examples to practice

- Prove that 2^n < n! for all integers n ≥ 4.
- Prove that 3 | (n^3 – n) for all positive integers n.
- Prove that $7^n - 2^n$ is divisible by 5 for all positive integers n.

Example 1

- <u>Prove that $7^n - 2^n$ is divisible by 5 for all positive integers n.</u>

Let P(k) be the propositional function "$7^k - 2^k$ is divisible by 5 ."

Prove that $\forall\, k \in Z^+, P(k)$

1. Base case: Show that P(1) is true.

$$7^1 - 2^1 = 5 \; is\ divisible\ by\ 5$$ **(True)**

2. Inductive step: Show that if P(n) is true, then P(n + 1) is true.

Assume that $7^n - 2^n$ is divisible by 5 is true.

Therefore, by the Inductive Hypothesis: $7^n - 2^n$ = 5s for some integer s. (7^n = 5s + 2^n)

We need to show that P(n + 1) is true, i.e.

Show that: $7^{n+1} - 2^{n+1}$ is divisible by 5

3) Don't forget the Conclusion

We start from:

$$7^{n+1} - 2^{n+1} = 7^n \times 7 - 2^n \times 2 = (5s + 2^n) \times 7 - 2^n \times 2 = 5 \times 7s + 7 \times 2^n - 2^n \times 2 = 5 \times 7s + 5 \times 2^n = 5 \times (7s + 2^n) = 5r$$

Example 2

- For all n ≥ 1, the sum of the first *n* even numbers equals $\text{n} \times (n+1)$.

  Let P(n) be the propositional function "the sum of the first *n* even numbers equals $\text{n} \times (n+1)$."

  We want to prove $\forall n \in Z^+$, P(n) where:

  $$P(n) = \sum_{i=1}^{n} 2i = n \times (n+1).$$

  1. Base case: Show that P(1) is true.

     $$2 = 1(1+1) = 2 \quad \textbf{(True)}$$

  2. Inductive step: Show that if P(k) is true, then P(k + 1) is true.

     Assume that $P(k) = \sum_{i=1}^{k} 2i = k \times (k+1)$.
     We need to show that P(k + 1) is true, i.e.
     Show that: $P(k) = \sum_{i=1}^{k+1} 2i = (k+1) \times (k+2)$.

$$\sum_{i=1}^{k+1} 2i = \sum_{i=1}^{k} 2i + 2(k+1) = k \times (k+1) + 2(k+1)$$

$$= (k+1) \times (k+2)$$

3) Don't forget the Conclusion

Strengthening the Inductive Hypothesis

- Suppose we want to prove $\forall \boldsymbol{x} \in \boldsymbol{Z}^+, \boldsymbol{P}(\boldsymbol{x})$ but proof doesn't go through.

- **Common trick:** Prove a stronger property Q(x)

- If $\forall x \in Z^+, Q(x) \rightarrow P(x)$ and $\forall x \in Z^+, Q(x)$ is provable, this implies $\forall \boldsymbol{x} \in \boldsymbol{Z}^+, \boldsymbol{P}(\boldsymbol{x})$.

- In many situations, strengthening inductive hypothesis allows proof to go through!

Example 3

- Prove the following theorem: "For all n $\geq$ 1, the sum of the first *n* odd numbers is a perfect square."

Definition: An integer *n* is called a **perfect square** if, and only if, $n = k^2$ for some integer *k*.

We want to prove $\forall n \in Z^+$, P(n) where:

$$P(n) = \sum_{i=1}^{n}(2i - 1) = k^2 \text{ for some integer k.}$$

Try to prove this using induction...

Example 3 (Cont.)

- Prove the following theorem: "For all n ≥ 1, the sum of the first *n* odd numbers is a perfect square."

- Let's use a stronger predicate:

$$\mathrm{Q}(n) = \sum_{i=1}^{n}(2i-1) = n^2.$$

- Clearly, $Q(n) \to P(n)$

Now prove $\forall n \in Z^+, \mathrm{Q(n)}$ using induction!

Example 4 (Hard)

Prove that $\sum_{k=1}^{2n} k = 2n^2 + n$.

Proof:

- $\sum_{i=1}^{n} 2i = n \times (n+1)$. (For even) -> From Example 2
- $\sum_{i=1}^{n} (2i-1) = n^2$. (For odd) -> From Example 3

Any two consecutive integers have opposite parity.

$\sum_{i=1}^{n} 2i + \sum_{i=1}^{n} (2i-1) = 1 + 2 + 3 + \cdots + 2n = \sum_{k=1}^{2n} k$

$$\sum_{k=1}^{2n} k = n^2 + n \times (n+1) = 2n^2 + n.$$

Is there any other way to prove it?

Use Arithmetic Series Formula.

Induction - Cardinality of Power Sets

- For all integers $n \geq 0$, if a set X has n elements, then $P(X)$ has 2^n elements.

<u>Proof (By Mathematical Induction)</u>

- Let the property $P(n)$ be the sentence "Any set with n elements has 2^n subsets."
- **Base case:** For n=0.

  Any set with 0 elements has $2^0 = 1$ subsets ({∅}) (True).
- **Inductive step:** *Show that for all integers $k \geq 0$, if $P(k)$ is true then $P(k + 1)$ is also true.*

  Assume that k is any integer with $k \geq 0$ such that:

  Any set with k elements has 2^k subsets.

Induction - Cardinality of Power Sets

We must show that P(k + 1) is true. That is:

Any set with k + 1 elements has 2^{k+1} subsets.

- Let *X* be a set with *k* + 1 elements. Since *k* + 1 ≥ 1, we may pick an element *z* in *X*.
- Observe that any subset of *X* either contains *z* or not. Furthermore, any subset of *X* that does not contain *z* is a subset of *X* − {*z*}.
- And any subset *A* of *X* − {*z*} can be matched up with a subset *B*, equal to *A* ∪ {*z*}, of *X* that contains *z*.
- Consequently, there are as many subsets of *X* that contain *z* as do not. Thus there are twice as many subsets of *X* as there are subsets of *X* − {*z*}.

But *X* − {*z*} has *k* elements, and so

the number of subsets of *X* − {*z*} = 2^k (Inductive Hypothesis)

the number of subsets of *X* = 2· (the number of subsets of *X* − {*z*})= $2 \times 2^k = 2^{k+1}$

Strong Mathematical Induction

Let *P(n)* be a property that is defined for integers *n*, and let *a* and *b* be fixed integers with $a \leq b$. Suppose the following two statements are true:

1. *P(a), P(a* + 1*), . . . ,* and *P(b)* are all true. **(basis step)**

2. For any integer $k \geq b$, if *P(i)* is true for all integers *i* from *a* through *k*, then show that *P(k* + 1*)* is true. **(inductive step)**

- If you can prove (1) and (2), you can **conclude**:

  for all integers $n \geq a$, $\boldsymbol{P(x)}$.

Example 5

Prove that any integer greater than 1 can be written as the product of primes.

Let the property *P(n)* be *"n* can be written as the product of primes".

Base case: Show that *P(2)* is true.

2 is the product of one prime: itself!

Inductive step: Show that for all integers k ≥ 2, if P(i) is true for all integers i from 2 through k, then P(k + 1) is also true:

- Inductive Hypothesis: Let *k* be any integer with *k* ≥ 2 and suppose that *i* can be written as the product of primes, for all integers *i* from 2 through *k.*

Example 5 (Cont.)

Therefore, we must show that k + 1 can be written as the product of primes.

- **Case 1 (k+1 is prime):** Then obviously P(k + 1) is true.
- **Case 2 (k+1 is composite):** In this case $k + 1 = a \times b$ where *a* and *b* are integers with $1 < a < k + 1$ and $1 < b < k + 1$. Thus, in particular, $2 \leq a \leq b \leq k$.

  By the **induction hypothesis**, both a and b can be written as the product of primes.

  Therefore, $k + 1 = a \times b$ can be written as the product of primes.

➢ We have shown that **every integer greater than 1** can be written as the product of primes.