Notes on Quantum Information Theory©

Samuel J. Lomonaco, Jr.*

November 19, 1996

Contents

1 Introduction 1
2 Some basic facts about QM systems 2
3 The partial trace 2
4 Classical Shannon entropy 3
5 Sundry formulas for the Shannon entropy 4
6 Von Neumann Entropy 4
7 Sundry formulas for the Von Neumann entropy 5
8 $H(A)$ versus $S(A)$? What is different? 6
9 Case I (Classical) Independent qubits 6
10 Case II (Classical) Correlated qubits 7
11 Case III (Nonclassical–Purely Quantum Mechanical) Entangled (supercorrelated) qubits, i.e., a pure ensemble of EPR pairs 8
12 Von Neumann entropy 9
13 Case I. Independent qubits 9
14 Case II. Classically correlated qubits 9
15 Case III. Entangled (supercorrelated) qubits 10
16 Summary of cases I, II, & III 11

1. Introduction

We consider different kinds of ensembles of qubit pairs.

Let $|i\rangle_A$ and $|j\rangle_B$ denote respectively the states of qubits A and B, lying respectively in the two dimensional Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, where $i = 0, 1$ and $j = 0, 1$. The n the ket

$$|i\rangle_A \otimes |j\rangle_B$$

representing the state of the qubit pair lies in the Hilbert space $\mathcal{H}_1 \otimes \mathcal{H}_2$. We abbreviate this as

$$|i\rangle_A \otimes |j\rangle_B = |i\rangle\,|j\rangle = |ij\rangle$$

*© Copyright 1996

†Partially supported by the L-O-O-P Fund.

We also write the adjoint of $|ij\rangle$ as

$$|ij\rangle^{\dagger} = \langle ji|$$

2. SOME BASIC FACTS ABOUT QM SYSTEMS

Principle 1[1]**.** Given QM systems $\mathcal{S}_1$ and $\mathcal{S}_2$ represented by kets $|\Psi_1\rangle \in \mathcal{H}_1$ and $|\Psi_2\rangle \in \mathcal{H}_2$, the global QM system $\mathcal{S}_1\mathcal{S}_2$ obtained by considering them together is represented by the ket

$$|\Psi_1\rangle \otimes |\Psi_2\rangle \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

P**rinciple 2.** Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be two different QM systems with state spaces $\mathcal{H}_1$ and $\mathcal{H}_2$, repectively. Then the state space of the global system $\mathcal{S}_1\mathcal{S}_2$ is $\mathcal{H}_1 \otimes \mathcal{H}_2$. The density operator ρ_1 of the system $\mathcal{S}_1$ alone (ignoring the system $\mathcal{S}_2$) is given by the partial trace

$$\rho_1 = Tr_2(\rho).$$

In like manner, the density operator ρ_2 of the system $\mathcal{S}_2$ is

$$\rho_2 = Tr_1(\rho).$$

3. THE PARTIAL TRACE

Let

$$\begin{cases} d_1 = Dim\,(\mathcal{H}_1) \\ \\ d_2 = Dim\,(\mathcal{H}_2) \end{cases}$$

The partial trace is defined as follows:

$$\begin{array}{ccc} & \overset{\text{Basis } \{e_i \otimes f_j\}}{\underset{\Phi}{\longrightarrow}} & \\ Hom\,(\mathcal{H}_1 \otimes \mathcal{H}_2,\ \mathcal{H}_1 \otimes \mathcal{H}_2) & & \mathcal{M}_{d_1 d_2}(\mathbb{C}) \\ \downarrow Tr_2 & & \downarrow Tr_2 \\ & \overset{\widetilde{\Phi}}{\underset{\text{Basis } \{e_i\}}{\longrightarrow}} & \\ Hom\,(\mathcal{H}_1,\ \mathcal{H}_1) & & \mathcal{M}_{d_1}(\mathbb{C}) \end{array}$$

[1]Please note that the converse of this principle is more subtle, for example for entangled systems.

Thus for a particular F, we have

$$\begin{array}{ccc} F & \overset{\Phi}{\longmapsto} & F_{ij,pq} \\ \downarrow Tr_2 & & \downarrow Tr_2 \\ Tr_2(F) & \overset{\widetilde{\Phi}}{\longmapsto} & \sum_j F_{ij,pj} \end{array}$$

The partial trace $Tr_2(F)$ of an operator $F \in Hom(\mathcal{H}_1 \otimes \mathcal{H}_2,\ \mathcal{H}_1 \otimes \mathcal{H}_2)$ is defined as

$$Tr_2(F) = \widetilde{\Phi}^{-1}(Tr_2(\Phi(F))),$$

where Φ and $\widetilde{\Phi}$ depend on the bases $\{e_i\}$ and $\{f_i\}$ chosen for $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively. It can be shown that $Tr_2(F)$ is independent of the bases chosen for $\mathcal{H}_1$ and $\mathcal{H}_2$.

Please also note that

$$Tr(F) = Tr_1(Tr_2(F))$$

4. Classical Shannon entropy

Consider a source A producing symbols from a set $A = \{a_1,\ a_2,\ \dots\}$, with each a_j occuring with probability p_j. (Hence, $\sum_j p_j = 1$.) Then the Shannon entropy of A, written $H(A)$, is defined as

$$H(A) = -\sum_i p_i \lg p_i$$

$H(A)$ ... The uncertainty of A

$$H(A) = -\sum_i p_i \lg p_i$$

$H(AB)$ ... The uncertainty of AB

$$H(AB) = -\sum_{i,j} p_{ij} \lg p_{ij}$$

$H(A|B)$ ... The uncertainty of A given B (Conditional entropy)

$$H(A|B) = -\sum_{i,j} p_{ij} \lg p_{i|j},$$

where

$$p_{i|j} = \frac{p_{ij}}{p_j} = \text{ Probability of } i \text{ given } j$$

$H(A : B)$ ... The uncertanty common to (shared by) both A and B. (Mutual entropy)

$$H(A : B) = -\sum_{i,j} p_{ij} \lg p_{i:j},$$

where

$$p_{i:j} = \frac{p_i p_j}{p_{ij}}$$

5. Sundry formulas for the Shannon entropy

$$\begin{cases} H(AB) = H(A) + H(B|A) = H(B) + H(A|B) \\ H(A : B) = H(A) + H(B) - H(AB) \end{cases}$$

$$\begin{cases} H(AB) = H(A|B) + H(A : B) + H(B|A) \\ H(A) = H(A|B) + H(A : B) \\ H(B) = H(B|A) + H(A : B) \end{cases}$$

6. Von Neumann Entropy

Let A be a quantum mechanical system with density operator ρ. Then the **Von Neumann entropy**, written $S(\rho)$, is defined as

$$S(A) = -Tr\,(\rho \lg \rho)$$

Remark 1. *Please not that the Von Neuman entropy is an invariant of a closed quantum mechanical system, i.e., invariant under unitary transformations. On the other hand, the Shannon entropy is not. Hence, the Von Neumann entropy is the true entropy of the system.*

Let ρ_A, ρ_B, and ρ_{AB}, be the density operators of the QM systems A, B, and AB. Then

$$\begin{aligned} \rho_A &= Tr_B\left(\rho_{AB}\right) \\ &\text{and} \\ \rho_B &= Tr_A\left(\rho_{AB}\right) \end{aligned}$$

$S(A)$ ... The uncertainty of A

$$S(A) = -Tr\left(\rho_A \lg \rho_A\right)$$

$S(AB)$ ... The uncertainty of AB

$$S(AB) = -Tr\left(\rho_{AB} \lg \rho_{AB}\right)$$

$S(A|B)$ ... The uncertainty of A given B (Conditional entropy)

$$S(A|B) = -Tr\left(\rho_{AB} \lg \rho_{A|B}\right),$$

where

$$\rho_{A|B} = \lim_{n\to\infty}\left[\rho_{AB}^{\frac{1}{n}}\left(1_A \otimes \rho_B\right)^{-\frac{1}{n}}\right]^n \approx \rho_{AB}\left(1_A \otimes \rho_B\right)^{-1}$$

$S(A:B)$ ... The uncertanty common to (shared by) both A and B. (Mutual entropy)

$$S(A:B) = -Tr\left(\rho_{AB} \lg \rho_{A:B}\right),$$

where

$$\rho_{A:B} = \lim_{n\to\infty}\left[\left(\rho_A \otimes \rho_B\right)^{\frac{1}{n}} \rho_{AB}^{-\frac{1}{n}}\right]^n \approx \left(\rho_A \otimes \rho_B\right)\rho_{AB}^{-1}$$

7. SUNDRY FORMULAS FOR THE VON NEUMANN ENTROPY

$$\begin{cases} S(AB) = S(A) + S(B|A) = S(B) + S(A|B) \\ \\ S(A:B) = S(A) + S(B) - S(AB) \end{cases}$$

$$\begin{cases} S(AB) = S(A|B) + S(A:B) + S(B|A) \\ S(A) = S(A|B) + S(A:B) \\ S(B) = S(B|A) + S(A:B) \end{cases}$$

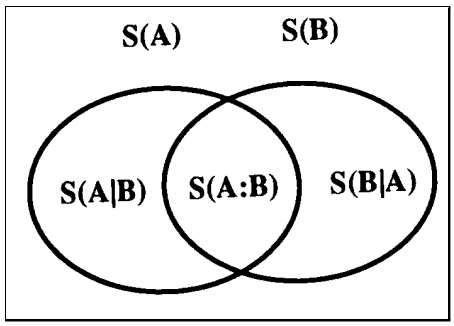


Figure1. General entropy diagram for a composite system AB.

8. $H(A)$ versus $S(A)$? What is different?

$$H(A:B) \leq \min\left[H(A), H(B)\right]$$

But,

$$S(A:B) \leq 2\min\left[S(A), S(B)\right]$$

$S(A:B)$ measures QM entanglement (supercorrelation) as well as classical correlation. $H(A:B)$ only measures classical correlation. As a result, $S(A:B)$ can become negative for entangled QM systems!

9. Case I (Classical) Independent qubits

We consider the following mixed ensemble

States	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$	All in $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

The corresponding density operator ρ is

$$\rho = \frac{1}{4}\left(|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11|\right)$$

The corresponding density matrix with respect to the basis

$$|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle$$

is

$$\begin{array}{c} \\ \langle 00| \\ \langle 10| \\ \langle 01| \\ \langle 11| \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \end{array} = \Delta\left(\tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4},\ \tfrac{1}{4}\right)$$

10. Case II (Classical) Correlated qubits

We consider the following mixed ensemble

States	$\|00\rangle$	$\|11\rangle$	All in $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	

The corresponding density operator is

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

The corresponding density matrix with respect to the basis

$$|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle$$

is

$$\begin{array}{c} \\ \langle 00| \\ \langle 10| \\ \langle 01| \\ \langle 11| \end{array} \begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\ \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{array} = \Delta\left(\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{2}\right)$$

11. CASE III (NONCLASSICAL–PURELY QUANTUM MECHANICAL) ENTANGLED (SUPERCORRELATED) QUBITS, I.E., A PURE ENSEMBLE OF EPR PAIRS

We consider the pure ensemble given by the state ket

$$|\Psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

The corresponding density operator ρ is

$$\begin{aligned}
\rho &= |\Psi\rangle\langle\Psi| \\
&= \left[\frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)\right]\left[\frac{1}{\sqrt{2}}\left(\langle 00| + \langle 11|\right)\right] \\
&= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)
\end{aligned}$$

The corresponding density matrix with respect to the basis

$$|00\rangle\,,\ |01\rangle\,,\ |10\rangle\,,\ |11\rangle$$

is

$$\begin{array}{c} \\ \langle 00| \\ \langle 10| \\ \langle 01| \\ \langle 11| \end{array}
\begin{array}{c}
\begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\
\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix}
\end{array}$$

Please note that $|\Psi\rangle$ can be extended to an orthonormal basis

$$|\Psi\rangle\,,\ \left|\Psi_1^{\perp}\right\rangle,\ \left|\Psi_2^{\perp}\right\rangle,\ \left|\Psi_3^{\perp}\right\rangle$$

of $\mathcal{H}_1 \otimes \mathcal{H}_2$. With respect to this basis, the matrix for the density operator ρ becomes

$$\begin{array}{c} \\ \langle\Psi\ | \\ \left\langle\Psi_1^{\perp}\right| \\ \left\langle\Psi_2^{\perp}\right| \\ \left\langle\Psi_3^{\perp}\right| \end{array}
\begin{array}{c}
\begin{array}{cccc} |\Psi\rangle & \left|\Psi_1^{\perp}\right\rangle & \left|\Psi_2^{\perp}\right\rangle & \left|\Psi_3^{\perp}\right\rangle \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}
\end{array}
= \Delta\,(1,\ 0,\ 0,\ 0)$$

12. VON NEUMANN ENTROPY

We now will consider the **Von Neumann entropy**

$$S(\rho) = -Tr\,(\rho \lg \rho)$$

for the qubit ensembles given in cases I, II, and III above. In each of the calculations given below, we repeatedly use the observation that

$$\lg\left(\Delta(d_1,\ d_2,\ d_3,\ d_4\right) = \Delta(\lg d_1,\ \lg d_2,\ \lg d_3,\ \lg d_4)$$

13. CASE I. INDEPENDENT QUBITS

We compute in the basis

$$|00\rangle\,,\ |01\rangle\,,\ |10\rangle\,,\ |11\rangle$$

Thus,

$$\begin{aligned}
S(\rho) &= -Tr\,(\rho \lg \rho) \\
&= -Tr\left[\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \lg \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}\right] \\
&= -Tr\left[\Delta\left(\tfrac{1}{4}\lg(\tfrac{1}{4}),\ \tfrac{1}{4}\lg(\tfrac{1}{4}),\ \tfrac{1}{4}\lg(\tfrac{1}{4}),\ \tfrac{1}{4}\lg(\tfrac{1}{4}),\ \right)\right] \\
&= -4\left(\tfrac{1}{4}\lg(\tfrac{1}{4})\right) = 2 \text{ qubits of uncertainty}
\end{aligned}$$

14. CASE II. CLASSICALLY CORRELATED QUBITS

We compute in the basis

$$|00\rangle\,,\ |01\rangle\,,\ |10\rangle\,,\ |11\rangle$$

Thus,

$$\begin{aligned} S(\rho) &= -Tr\left(\rho \lg \rho\right) \\ &= -Tr\left[\Delta\left(\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{2}\right) \lg \Delta\left(\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{2}\right)\right] \\ &= -Tr\left[\Delta\left(\tfrac{1}{2}\lg\tfrac{1}{2},\ 0,\ 0,\ \tfrac{1}{2}\lg\tfrac{1}{2}\right)\right] \\ &= -2\left[\tfrac{1}{2}\lg\tfrac{1}{2}\right] \\ &= 1 \text{ qubit of uncertainty} \end{aligned}$$

15. Case III. Entangled (supercorrelated) qubits

We compute in the basis

$$|\Psi\rangle,\ \left|\Psi_1^{\perp}\right\rangle,\ \left|\Psi_2^{\perp}\right\rangle,\ \left|\Psi_3^{\perp}\right\rangle$$

Thus,

$$\begin{aligned} S(\rho) &= -Tr\left(\rho \lg \rho\right) \\ &= -Tr\left[\Delta\left(1,\ 0,\ 0,\ 0\right) \lg \Delta\left(1,\ 0,\ 0,\ 0\right)\right] \\ &= -Tr\left[\Delta\left(1 \cdot \lg 1,\ 0,\ 0,\ 0\right)\right] \\ &= 0 \text{ qubits of uncertainty} \end{aligned}$$

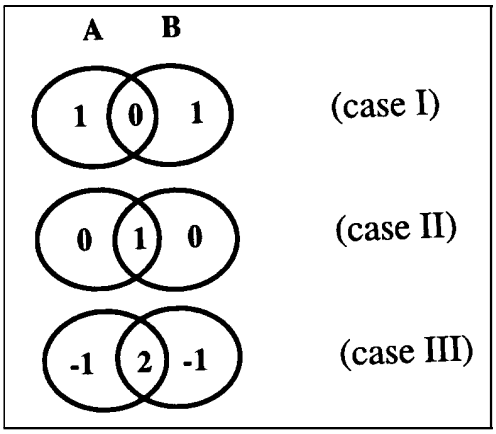
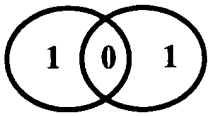


Figure 2. Entropy diagrams for cases I, II, and III.

16. SUMMARY OF CASES I, II, & III

Case I. (Classical) Independent Qubits. $S(AB) = 2$ & $S(A) = 1 = S(B)$



(case I)

$$\begin{cases} \rho_A &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow & \Delta\left(\frac{1}{2},\frac{1}{2}\right) \\ \rho_B &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow & \Delta\left(\frac{1}{2},\frac{1}{2}\right) \\ \rho_{AB} &= \frac{1}{4}|00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| & \leftrightarrow & \Delta\left(\frac{1}{4},\frac{1}{4},\frac{1}{4},\frac{1}{4}\right) \end{cases}$$

A

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_1$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

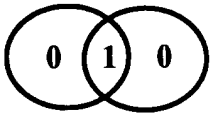
B

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB

States	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$	All in $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	Mixed Ensemble

Case II. **(Classical) Correlated Qubits.** $S(AB) = 1$ & $S(A) = 1 = S(B)$



(case II)

$$\begin{cases} \rho_A &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow & \Delta\left(\frac{1}{2},\frac{1}{2}\right) \\ \rho_B &= \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow & \Delta\left(\frac{1}{2},\frac{1}{2}\right) \\ \rho_{AB} &= \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) & \leftrightarrow & \Delta\left(\frac{1}{2},0,0,\frac{1}{2}\right) \end{cases}.$$

A

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_1$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

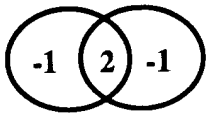
B

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB

States	$\|00\rangle$	$\|11\rangle$	All in $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

Case III. **(Nonclassical-Purely QM) Entangled (Supercorrelated) Qubits.** $S(AB) = 0$ & $S(A) = 1 = S(B)$



(case III)

$$\begin{cases} \rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \leftrightarrow \Delta\left(\frac{1}{2}, \frac{1}{2}\right) \\ \rho_B = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) \leftrightarrow \Delta\left(\frac{1}{2}, \frac{1}{2}\right) \\ \rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right) \leftrightarrow \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{cases}$$

A

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_1$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

B

States	$\|0\rangle$	$\|1\rangle$	All in $\mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB

States	$\frac{1}{\sqrt{2}}\left(\|00\rangle + \|11\rangle\right)$	In $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	1	Pure Ensemble

REFERENCES

[1] Cerf, Nicholas J., and Chris Adami, **Quantum mechanics of measurement**, quant-ph/9605002, 2 May 1996. (Submitted to Physical Review A).