Notes on Quantum Information Theory [*]

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Contents

1. Introduction

We consider different kinds of ensembles of qubit pairs.

Let $|i\rangle_A$ and $|j\rangle_B$ denote respectively the states of qubits A and B, lying respectively in the two dimensional Hilbert spaces H_1 and H_2, where $i = 0, 1$ and $j = 0, 1$. Then the ket

$$|i\rangle_A \otimes |j\rangle_B$$

representing the state of the qubit pair lies in the Hilbert space $H_1 \otimes H_2$. We abbreviate this as

$$|i\rangle_A \otimes |j\rangle_B = |i\rangle|j\rangle = |ij\rangle$$

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We also write the adjoint of jiji as

$$\text{jiji}^{y} = \text{hjij}$$

2. Some basic facts about QM systems

Principle 1[1]. Given QM systems S_1 and S_2 represented by kets $\text{j}^{\text{a}}{}_1\text{i}$ 2 H_1 and $\text{j}^{\text{a}}{}_2\text{i}$ 2 H_2, the global QM system S_1S_2 obtained by considering them together is represented by the ket

$$\text{j}^{\text{a}}{}_1\text{i} - \text{j}^{\text{a}}{}_2\text{i} \; 2 \; H_1 - H_2$$

Principle 2. Let S_1 and S_2 be two di¤erent QM systems with state spaces H_1 and H_2, repectively. Then the state space of the global system S_1S_2 is $H_1 - H_2$. The density operator ½$_1$ of the system S_1 alone (ignoring the system S_2) is given by the partial trace

$$½_1 = Tr_2(½):$$

In like manner, the density operator ½$_2$ of the system S_2 is

$$½_2 = Tr_1(½):$$

3. The partial trace

Let

$$\begin{array}{l} 8 \\ < \; d_1 = Dim(H_1) \\ : \; d_2 = Dim(H_2) \end{array}$$

The partial trace is de…ned as follows:

$$\begin{array}{ccc} & \text{Basis fe}_i - f_j\text{g} & \\ Hom(H_1 - H_2;\ H_1 - H_2) & ¡! & M_{d_1d_2}(C) \\ & © & \\ \#\,Tr_2 & & \#\,Tr_2 \\ & \text{e} & \\ Hom(H_1;\ H_1) & ¡! & M_{d_1}(C) \\ & \text{Basis fe}_i\text{g} & \end{array}$$

[1] Please note that the converse of this principle is more subtle, for example for entangled systems.

Thus for a particular F, we have

$$\begin{array}{ccc} F & \overset{\Psi}{\longmapsto} & F_{ij;pq} \\ \downarrow Tr_2 & & \downarrow Tr_2 \\ Tr_2(F) & \overset{\Phi}{\longmapsto} & \sum_j F_{ij;pj} \end{array}$$

The partial trace $Tr_2(F)$ of an operator $F \in Hom(H_1 \otimes H_2, H_1 \otimes H_2)$ is defined as

$$Tr_2(F) = \Phi^{-1}(Tr_2(\Psi(F))),$$

where Ψ and Φ depend on the bases $\{e_i\}$ and $\{f_i\}$ chosen for H_1 and H_2, respectively. It can be shown that $Tr_2(F)$ is independent of the bases chosen for H_1 and H_2.

Please also note that

$$Tr(F) = Tr_1(Tr_2(F))$$

4. Classical Shannon entropy

Consider a source A producing symbols from a set $A = \{a_1, a_2, \ldots\}$, with each a_j occuring with probability p_j. (Hence, $\sum_j p_j = 1$.) Then the Shannon entropy of A, written $H(A)$, is defined as

$$H(A) = -\sum_i p_i \lg p_i$$

$H(A)$ ... The uncertainty of A

$$H(A) = -\sum_i p_i \lg p_i$$

$H(AB)$ ... The uncertainty of AB

$$H(AB) = -\sum_{i,j} p_{ij} \lg p_{ij}$$

$H(A|B)$ $\cdots$ The uncertainty of A given B (Conditional entropy)

$$H(A|B) = -\sum_{i,j} p_{ij} \lg p_{i|j},$$

where

$$p_{i|j} = \frac{p_{ij}}{p_j} = \text{ Probability of } i \text{ given } j$$

$H(A:B)$ $\cdots$ The uncertanty common to (shared by) both A and B. (Mutual entropy)

$$H(A:B) = -\sum_{i,j} p_{ij} \lg p_{i:j},$$

where

$$p_{i:j} = \frac{p_i p_j}{p_{ij}}$$

5. Sundry formulas for the Shannon entropy

$$\begin{cases} H(AB) = H(A) + H(B|A) = H(B) + H(A|B) \\ H(A:B) = H(A) + H(B) - H(AB) \end{cases}$$

$$\begin{cases} H(AB) = H(A|B) + H(A:B) + H(B|A) \\ H(A) = H(A|B) + H(A:B) \\ H(B) = H(B|A) + H(A:B) \end{cases}$$

6. Von Neumann Entropy

Let A be a quantum mechanical system with density operator ρ. Then the Von Neumann entropy, written $S(\rho)$, is defined as

$$S(A) = -Tr(\rho \lg \rho)$$

Remark 1. Please not that the Von Neuman entropy is an invariant of a closed quantum mechanical system, i.e., invariant under unitary transformations. On the other hand, the Shannon entropy is not. Hence, the Von Neumann entropy is the true entropy of the system.

Let ρ_A, ρ_B, and ρ_{AB}, be the density operators of the QM systems A, B, and AB. Then

$$
\begin{aligned}
\rho_A &= Tr_B\left(\rho_{AB}\right) \\
&\text{and} \\
\rho_B &= Tr_A\left(\rho_{AB}\right)
\end{aligned}
$$

$S(A)$ ::: The uncertainty of A

$$S(A) = -Tr\left(\rho_A \lg \rho_A\right)$$

$S(AB)$ ::: The uncertainty of AB

$$S(AB) = -Tr\left(\rho_{AB} \lg \rho_{AB}\right)$$

$S(A|B)$ ::: The uncertainty of A given B (Conditional entropy)

$$S(A|B) = -Tr\left(\rho_{AB} \lg \rho_{A|B}\right),$$

where

$$\rho_{A|B} = \lim_{n \to \infty} \left[\rho_{AB}^{\frac{1}{n}} \left(1_A \otimes \rho_B\right)^{-\frac{1}{n}}\right]^n \simeq \rho_{AB}\left(1_A \otimes \rho_B\right)^{-1}$$

$S(A:B)$ ::: The uncertanty common to (shared by) both A and B. (Mutual entropy)

$$S(A:B) = -Tr\left(\rho_{AB} \lg \rho_{A:B}\right),$$

where

$$\rho_{A:B} = \lim_{n \to \infty} \left[\left(\rho_A \otimes \rho_B\right)^{\frac{1}{n}} \rho_{AB}^{-\frac{1}{n}}\right]^n \simeq \left(\rho_A \otimes \rho_B\right) \rho_{AB}^{-1}$$

7. Sundry formulas for the Von Neumann entropy

$$
\left\{
\begin{aligned}
&S(AB) = S(A) + S(B|A) = S(B) + S(A|B) \\
&S(A:B) = S(A) + S(B) - S(AB)
\end{aligned}
\right.
$$

$$
\left\{
\begin{aligned}
&S(AB) = S(A|B) + S(A:B) + S(B|A) \\
&S(A) = S(A|B) + S(A:B) \\
&S(B) = S(B|A) + S(A:B)
\end{aligned}
\right.
$$

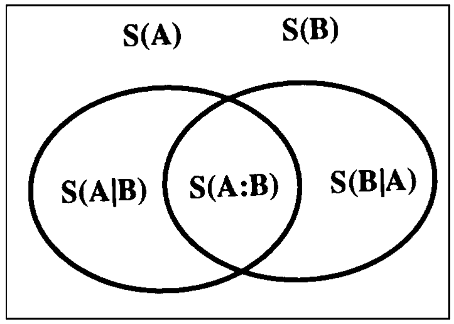


Figure1. General entropy diagram for a composite system AB.

8. H(A) versus S(A)? What is different?

$$H(A:B) \quad \min[H(A);H(B)]$$

But,

$$S(A:B) \quad 2\min[S(A);S(B)]$$

S(A : B) measures QM entanglement (supercorrelation) as well as classical correlation. H(A : B) only measures classical correlation. As a result, S(A : B) can become negative for entangled QM systems!

9. Case I (Classical) Independent qubits

We consider the following mixed ensemble

States	j00i	j01i	j10i	j11i	All in H_1 - H_2
Freq	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	

The corresponding density operator ½ is

$$½ = \frac{1}{4}(\text{j00i h00j} + \text{j01i h10j} + \text{j10i h01j} + \text{j11i h11j})$$

The corresponding density matrix with respect to the basis

$$\text{j00i ; j01i ; j10i ; j11i}$$

is

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline \langle 00| & \frac{1}{4} & 0 & 0 & 0 \\ \langle 10| & 0 & \frac{1}{4} & 0 & 0 \\ \langle 01| & 0 & 0 & \frac{1}{4} & 0 \\ \langle 11| & 0 & 0 & 0 & \frac{1}{4} \end{array} = \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)$$

10. Case II (Classical) Correlated qubits

We consider the following mixed ensemble

States	$\|00\rangle$	$\|11\rangle$	All in $\mathcal{H}_1 \otimes \mathcal{H}_2$
Freq	$\frac{1}{2}$	$\frac{1}{2}$	

The corresponding density operator is

$$\rho = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right)$$

The corresponding density matrix with respect to the basis

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

is

$$\begin{array}{c|cccc} & |00\rangle & |01\rangle & |10\rangle & |11\rangle \\ \hline \langle 00| & \frac{1}{2} & 0 & 0 & 0 \\ \langle 10| & 0 & 0 & 0 & 0 \\ \langle 01| & 0 & 0 & 0 & 0 \\ \langle 11| & 0 & 0 & 0 & \frac{1}{2} \end{array} = \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)$$

11. Case III (Nonclassical–Purely Quantum Mechanical) Entangled (supercorrelated) qubits, i.e., a pure ensemble of EPR pairs

We consider the pure ensemble given by the state ket

$$|\psi\rangle = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right) \in \mathcal{H}_1 \otimes \mathcal{H}_2$$

The corresponding density operator ρ is

$$\begin{aligned}
\rho &= |\psi\rangle\langle\psi| \\
&= \left(\frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)\right)\left(\frac{1}{\sqrt{2}}(\langle 00| + \langle 11|)\right) \\
&= \frac{1}{2}\left(|00\rangle\langle 00| + |00\rangle\langle 11| + |11\rangle\langle 00| + |11\rangle\langle 11|\right)
\end{aligned}$$

The corresponding density matrix with respect to the basis

$$|00\rangle,\ |01\rangle,\ |10\rangle,\ |11\rangle$$

is

$$\begin{array}{c} \\ \langle 00| \\ \langle 10| \\ \langle 01| \\ \langle 11| \end{array}
\begin{array}{c} \begin{array}{cccc} |00\rangle & |01\rangle & |10\rangle & |11\rangle \end{array} \\
\begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{array}$$

Please note that $|\psi\rangle$ can be extended to an orthonormal basis

$$|\psi\rangle,\ \left|\psi_1^\perp\right\rangle,\ \left|\psi_2^\perp\right\rangle,\ \left|\psi_3^\perp\right\rangle$$

of $H_1 \otimes H_2$. With respect to this basis, the matrix for the density operator ρ becomes

$$\begin{array}{c} \\ \langle\psi| \\ \left\langle\psi_1^\perp\right| \\ \left\langle\psi_2^\perp\right| \\ \left\langle\psi_3^\perp\right| \end{array}
\begin{array}{c} \begin{array}{cccc} |\psi\rangle & \left|\psi_1^\perp\right\rangle & \left|\psi_2^\perp\right\rangle & \left|\psi_3^\perp\right\rangle \end{array} \\
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \end{array} = \mathrm{diag}(1, 0, 0, 0)$$

12. Von Neumann entropy

We now will consider the Von Neumann entropy

$$S(\rho) = -\mathrm{Tr}(\rho \lg \rho)$$

for the qubit ensembles given in cases I, II, and III above. In each of the calculations given below, we repeatedly use the observation that

$$\lg\left(\Delta(d_1, d_2, d_3, d_4)\right) = \Delta(\lg d_1, \lg d_2, \lg d_3, \lg d_4)$$

13. Case I. Independent qubits

We compute in the basis

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

Thus,

$$\begin{aligned}
S(\rho) &= -\operatorname{Tr}(\rho \lg \rho) \\
&= -\operatorname{Tr}\left[\begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix} \lg \begin{pmatrix} \frac{1}{4} & 0 & 0 & 0 \\ 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{1}{4} & 0 \\ 0 & 0 & 0 & \frac{1}{4} \end{pmatrix}\right] \\
&= -\operatorname{Tr}\left[\Delta\left(\tfrac{1}{4}\lg(\tfrac{1}{4}), \tfrac{1}{4}\lg(\tfrac{1}{4}), \tfrac{1}{4}\lg(\tfrac{1}{4}), \tfrac{1}{4}\lg(\tfrac{1}{4})\right)\right] \\
&= -4\left(\tfrac{1}{4}\lg(\tfrac{1}{4})\right) = 2 \text{ qubits of uncertainty}
\end{aligned}$$

14. Case II. Classically correlated qubits

We compute in the basis

$$|00\rangle, |01\rangle, |10\rangle, |11\rangle$$

Thus,

$$\begin{aligned}
S(\rho) &= -\operatorname{Tr}(\rho \lg \rho) \\
&= -\operatorname{Tr}\left[\Delta\left(\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right) \lg \Delta\left(\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\right)\right] \\
&= -\operatorname{Tr}\left[\Delta\left(\tfrac{1}{2}\lg\tfrac{1}{2}, 0, 0, \tfrac{1}{2}\lg\tfrac{1}{2}\right)\right] \\
&= -2\left[\tfrac{1}{2}\lg\tfrac{1}{2}\right] \\
&= 1 \text{ qubit of uncertainty}
\end{aligned}$$

15. Case III. Entangled (supercorrelated) qubits

We compute in the basis

j ª i ; ¯¯ a ?$_1$ ® ; ¯¯ a ?$_2$ ® ; ¯¯ a ?$_3$ ®

Thus,

S(½) = ¡ Tr (½ lg ½)

= ¡ Tr [¢ (1; 0; 0; 0) lg ¢ (1; 0; 0; 0)]

= ¡ Tr [¢ (1 ¢ lg 1; 0; 0; 0)]

= 0 qubits of uncertainty

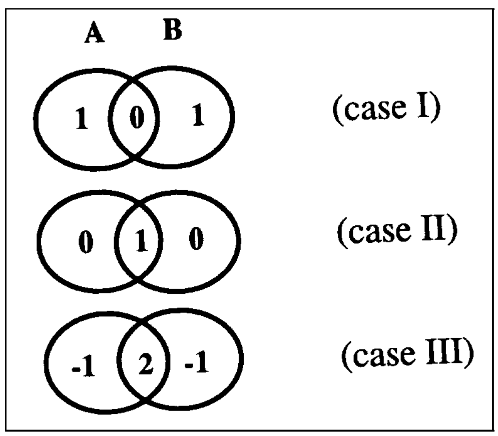
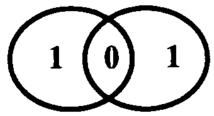


Figure 2. Entropy diagrams for cases I, II, and III.

16. Summary of cases I, II, & III

Case I. (Classical) Independent Qubits. S(AB) = 2 & S(A) = 1 = S(B)



(case I)

$$
\begin{cases}
\rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\
\rho_B = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\
\rho_{AB} = \frac{1}{4}\, |00\rangle\langle 00| + |01\rangle\langle 10| + |10\rangle\langle 01| + |11\rangle\langle 11| & \leftrightarrow \left(\frac{1}{4}, \frac{1}{4}, \frac{1}{4}, \frac{1}{4}\right)
\end{cases}
$$

A

States	$\|0\rangle$	$\|1\rangle$	All in H_1
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

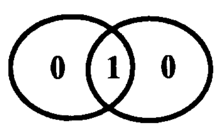
B

States	$\|0\rangle$	$\|1\rangle$	All in H_2
Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB

States	$\|00\rangle$	$\|01\rangle$	$\|10\rangle$	$\|11\rangle$	All in H_1 - H_2
Freq	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	$\frac{1}{4}$	Mixed Ensemble

Case II. (Classical) Correlated Qubits. S(AB) = 1 & S(A) = 1 = S(B)



(case II)

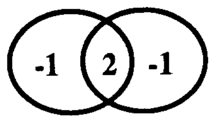
$$
\begin{cases}
\rho_A = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\
\rho_B = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) & \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\
\rho_{AB} = \frac{1}{2}\left(|00\rangle\langle 00| + |11\rangle\langle 11|\right) & \leftrightarrow \left(\frac{1}{2}, 0, 0, \frac{1}{2}\right)
\end{cases}
$$

A	States	$\lvert 0\rangle$	$\lvert 1\rangle$	All in H_1
	Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

B	States	$\lvert 0\rangle$	$\lvert 1\rangle$	All in H_2
	Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB	States	$\lvert 00\rangle$	$\lvert 11\rangle$	All in $H_1 - H_2$
	Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

Case III. (Nonclassical-Purely QM) Entangled (Supercorrelated) Qubits.
$S(AB) = 0$ & $S(A) = 1 = S(B)$



(case III)

$$\begin{cases} \rho_A = \frac{1}{2}(\lvert 0\rangle\langle 0\rvert + \lvert 1\rangle\langle 1\rvert) \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\ \rho_B = \frac{1}{2}(\lvert 0\rangle\langle 0\rvert + \lvert 1\rangle\langle 1\rvert) \leftrightarrow \left(\frac{1}{2}, \frac{1}{2}\right) \\ \rho_{AB} = \frac{1}{2}(\lvert 00\rangle\langle 00\rvert + \lvert 00\rangle\langle 11\rvert + \lvert 11\rangle\langle 00\rvert + \lvert 11\rangle\langle 11\rvert) \leftrightarrow \begin{pmatrix} \frac{1}{2} & 0 & 0 & \frac{1}{2} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ \frac{1}{2} & 0 & 0 & \frac{1}{2} \end{pmatrix} \end{cases}$$

A	States	$\lvert 0\rangle$	$\lvert 1\rangle$	All in H_1
	Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

B	States	$\lvert 0\rangle$	$\lvert 1\rangle$	All in H_2
	Freq	$\frac{1}{2}$	$\frac{1}{2}$	Mixed Ensemble

AB	States	$\frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$	In $H_1 - H_2$
	Freq	1	Pure Ensemble

References

[1] Cerf, Nicholas J., and Chris Adami, Quantum mechanics of measurement, quant-ph/9605002, 2 May 1996. (Submitted to Physical Review A).