THE MACWILLIAMS AND PLESS IDENTITIES: A SUMMARY

PROFESSOR SAMUEL J. LOMONACO

Abstract. We give a brief summary of the MacWilliams and Pless Identies

1. Summary

Given the weight distribution of a linear code V, the MacWilliams and Pless identities provide a means of determining the weight distribution of the corresponding dual linear code $V^{\perp}$. We describe below these identities only for binary linear codes.

Theorem 1 (MacWilliams)**.** *Let V be a binary linear $[n,k]$ code, and let $V^{\perp}$ be the corresponding binary linear $[n,n-k]$ orthogonal complement code. Let A_i and $A_i^{\perp}$ denote the number of vectors of Hamming weight i in V and $V^{\perp}$, respectively Finally, let $A(x)$ and $A^{\perp}(x)$ be the weight enumerator polynomials respectively defined by*

$$A(x)=\sum_{i=0}^{n} A_i x^i \qquad \text{and} \qquad A^{\perp}(x)=\sum_{i=0}^{n} A_i^{\perp} x^i \quad .$$

Then

$$2^k A^{\perp}(x)=(1+x)^n A\left(\frac{1-x}{1+x}\right)$$

Corollary 1 (Pless)**.** *Since*

$$\left(x\frac{d}{dx}\right)^j A^{\perp}(x)=\sum_{i=0}^{n} i^j A_i^{\perp} x^i \ ,$$

we have

$$\left.\left(x\frac{d}{dx}\right)^j A^{\perp}(x)\right|_{x=1}=\sum_{i=0}^{n} i^j A_i^{\perp} \ ,$$

Hence,

$$\begin{aligned}
\textstyle\sum_{i=0}^{n} A_i^{\perp} &= 2^{n-k} \\
\textstyle\sum_{i=0}^{n} iA_i^{\perp} &= 2^{n-k-1}\left(n-A_1\right) \\
\textstyle\sum_{i=0}^{n} i^2 A_i^{\perp} &= 2^{n-k-2}\left[n\left(n+1\right)-2nA_1+2A_2\right]
\end{aligned}$$

Date: October 9, 2006.

Example 1. *Let V be the binary linear $[5,3]$ code given by the generator matrix*

$$G = \begin{pmatrix} 1 & 0 & 0 & \vdots & 0 & 1 \\ 0 & 1 & 0 & \vdots & 1 & 0 \\ 0 & 0 & 1 & \vdots & 1 & 1 \end{pmatrix}$$

Since the orthogonal complement $V^{\perp}$ is of lower dimension $5-3=2$, we will compute the weight enumerator $A^{\perp}(x)$ of $V^{\perp}$, and use the MacWilliams identity to find the weight enererator of V, and hence the minimum distance d of V.

Since the generator matrix G is of the form $G = (\ I \mid P\ )$, we know that the parity check matrix of V is given by

$$H = \left(\ -P^T \mid I\ \right) = \begin{pmatrix} 0 & 1 & 1 & \vdots & 1 & 0 \\ 1 & 0 & 1 & \vdots & 0 & 1 \end{pmatrix}$$

Since the parity check matrix H of V is also the generator matrix of $V^{\perp}$, we can use H to enumerate the elements of $V^{\perp}$, i.e.,

$$V^{\perp} = \left\{ uH : u \in GF(2)^2 \right\} \ .$$

So with H, we can construct the following table enumerating the weights of $V^{\perp}$.

InfoWord	*CodeWord*	*Weight*
00	00000	0
01	01110	3
10	10101	3
11	11011	4

Let $A_i^{\perp}$ be the number of elements of $V^{\perp}$ of weight i. Then from the above table, we see that the values of $A_i^{\perp}$ are:

i	$A_i^{\perp}$
0	1
1	0
2	0
3	2
4	1
5	0

Hence, the weight enumerator polynomial $A^{\perp}(x) = \sum_{i=0}^{5} A_i^{\perp} x^i$ is given by

$$A^{\perp}(x) = 1 \cdot x^0 + 0 \cdot x^1 + 0 \cdot x^2 + 2 \cdot x^3 + 1 \cdot x^4 + 0 \cdot x^5 = 1 + 2x^3 + x^4$$

But from the MacWilliams identity, we know that

$$2^2 A(x) = (1+x)^5 \, A^{\perp}\left(\frac{1-x}{1+x}\right)$$

Thus,

$$4A(x) = (1+x)^5 \left[1 + 2\left(\frac{1-x}{1+x}\right)^3 + \left(\frac{1-x}{1+x}\right)^4 \right]$$

$$= (1+x)^5 + 2(1-x)^3(1+x)^2 + (1-x)^4(1+x)$$

$$= 4 + 8x^2 + 16x^3 + 4x^4$$

Thus,

$$A(x) = 1 + 2x^2 + 4x^3 + x^4$$

Hence, we have

i	A_i
0	1
1	0
2	2
3	4
4	1
5	0

where A_i denotes the number of elements of V of weight i. Thus, the minimum distance d of V is $d = 2$.

A HANDOUT FOR CMSC 442/653