CS762: Graph-Theoretic Algorithms
Lecture 28: Independent Set in Planar Graphs
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Abstract

We prove an efficient approximation algorithm for finding a maximum independent set in planar graphs, due to Baker [Bak94], which is based on using outer-planar graphs. We mention (without proof) some other results on seperators in planar graphs, and suggest further research.

1 Introduction

We already know that finding a maximal independent set in a planar graph is NP-hard. This naturally suggests the question of whether we can at least approximate this in an efficient manner. The answer is yes, and furthermore, we can obtain an arbitrarily good approximation in linear time, at the cost of increasing the constant. The astute optimist might notice that as the problem is discrete, there is an approximation threshold above which an approximation becomes constant: finding an independent set of size at least $\frac{n-1}{n}\alpha$ is guaranteed to be optimal. Of course, to achieve this would require making the constant exponential in n. More precisely, we give an algorithm that produces an independent set of size $\frac{k-1}{k}\alpha$ in $O(k8^k n)$ time. The key concept is splitting the graph into (overlapping) subgraphs each of which is k-outerplanar. Note that we are *not* assuming that the original graph is k-outerplanar; in fact, we will not make use of the level of outerplanarity of the graph, except to tacitly acknowledge the fact that it is finite, i.e., G is t-outerplanar for some $t \le n$.

2 Aside

We recall a hardness result shown earlier.

Theorem 1 *planar 3-SAT $\le$ planar VertexCover $\le$ planar IS* □

In fact finding an independent set in planar graphs remains hard even for (planar) boxicity-2 graphs.

Theorem 2 *planar IS $\le$ boxicity-2 IS*

Proof: Let G be a planar graph, and let G' be the graph obtained by twice subdividing each edge of G. That is for each edge $(u,v) \in E(G)$, we delete the edge (u,v), add vertices $x_{u,v}, y_{u,v}$, and add edges $(u, x_{u,v}), (x_{u,v}, y_{u,v}), (y_{u,v}, v)$ (see Figure 2).

Now take a weak visibility representation of G (recall from a previous class that this always exists for planar graphs). From this we construct a boxicity-2 representation of G' as follows. The

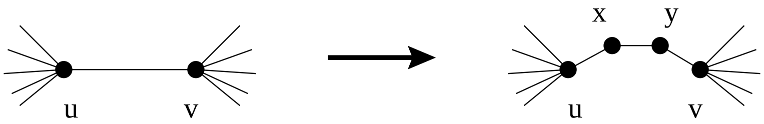

Figure 1: Subdividing G to get G'

boxes corresponding to vertices of G' that were already in G are just the boxes of these vertices in the visibility representation. For each edge in G, we know have a path of length 3 in G'. For each edge in G, there is a vertical line connecting the two boxes of the visibility representation of G that does not intersect any other two boxes. Replacing this line by two "thin vertical" boxes we obtain the boxicity-2 representation of G'. It is geometrically obvious (and simple but tedious to calculate) a value for the width of the new boxes such that they are guaranteed not to intersect any others.

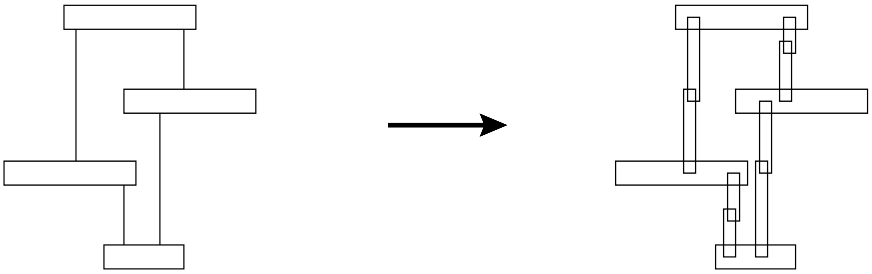

Figure 2: Obtaining a boxicity-2 representation of G' from a visibility representation of G

Recall a recent homework assignment that said that twice subdividing a single edge increases the size of a maximum independent set by exactly one. It is a simple generalization of the proof of this fact that twice subdividing t edges increases the size of a maximum independent set by exactly t, and thus $\alpha(G') = \alpha(G) + m(G)$. In fact, we can say slightly more (again based on the proof of the homework): given a maximum independent set for G we can easily obtain a maximum independent set for G', and the converse is true as well. Since computing $\alpha(G)$ is no harder than computing $\alpha(G')$, we have that computing an independent set in boxicity-2 graphs is NP-hard. □

3 Approximating a maximum independent set in a planar graph

Our main result is the following, due to Baker [Bak94].

Theorem 3 *For every planar graph G and every positive integer k, an independent set of size at least $\frac{k-1}{k}\alpha(G)$ can be found in $O(k8^k n)$ time.*

To prove this, we will make use of the ideas of k-outerplanar graphs. Let the original graph be t-outerplanar for some completely unknown t, and think informally of it as having t layers drawn in a concentric order (this is not quite true, of course, but it gives a useful picture). If we delete every k^{th} layer, then we obtain a series of "concentrically nested" subgraphs that are in fact disconnected – so they aren't really "concentric" at all. In fact, they are each $(k-1)$-outerplanar. We know how to efficiently find a maximum independent set for k-outerplanar graphs. Now the above tacitly assumed we deleted layers $k, 2k, 3k, \ldots$, but in fact we could have deleted layers

$j, k+j, 2k+j, 3k+j, \ldots$ for any $1 \leq j \leq k$. It is then a simple matter of choosing the optimum value of j. Thus we find maximum independent sets for subgraphs that are almost as big as the original graph and choose the best one.

Proof: First fix some (arbitrary) embedding of the graph $G = (V, E)$. Now let L_1 be the vertices of the outer face, and recursively define L_i to be the outer vertices of $G[V \setminus \{L_1 \cup \ldots \cup L_{i-1}\}]$. Note that although this process will certainly stop somewhere, we don't really care where. Note also that we keep the embedding fixed during the whole process, so that it is possible that we will at some point encounter, say, an outerplanar graph that is drawn without all vertices on the outer face; this will not fundamentally affect the result.

Now let k be some positive integer (chosen so as to give the desired approximation). For $1 \leq i \leq k$ we define the graphs

$$G_i = G[V \setminus \{L_i, L_{i+k}, L_{i+2k}, \ldots\}].$$

It is neither known or relevant how many layers L_j there are in all; the important point is that each L_j is in fact a separator of the graph. Consider a fixed G_i: it can be split into components

$$G[L_1 \cup \ldots \cup L_{i-1}], \;\; G[L_{i+1} \cup \ldots \cup L_{i+k-1}], \;\; G[L_{i+k+1} \cup \ldots \cup L_{i+2k-1}], \ldots$$

Now if we look at each component, then we see that it is $(k-1)$-outerplanar. In fact, they are by definition the "natural" $(k-1)$-outerplanar subgraphs of G with respect to the particular embedding, since we defined them by peeling off $k-1$ consecutive outer faces. Note that the components of G_i overlap those of G_j; this is not a problem, since we will consider the G_i separately.

We already know that a k-outerplanar graph is a partial $3k$-tree, and that we can obtain a maximum independent set of a partial k-tree in $O(2^k n)$ time. So that means we can find a maximum independent set in a particular G_i in $O(8^k n)$ time. So we can find the largest independent set contained in any of the G_i in $O(k8^k n)$ time.

It remains to show that the independent set we find is as big as we claim it is. To do this, consider a (true) maximum independent set I. Define $I_i = I \cap L_i$ for $i \geq 1$. Then

$$I = \dot{\bigcup}_{i \geq 1} I_i$$

Note that this is a disjoint union since the layers L_i form a partition of the vertex set of G. Now for $1 \leq i < k$ consider the sets

$$J_i = \bigcup_{j \geq 0} L_{i+jk}.$$

Note that J_i is exactly the set of vertices that was deleted from G to obtain G_i. Note also that

$$V(G) = \dot{\bigcup}_{i=1}^{k} J_i$$
$$I = \dot{\bigcup}_{i=1}^{k} (J_i \cap I)$$

which are also disjoint unions. Now we have the simple heart of the proof: a disjoint union of k sets must have at least one of its members of size at most the average. So let t be an integer such that $|J_t \cap I| \leq |I|/k$. This implies that $|I \setminus (J_t \cap I)| \geq \frac{k-1}{k}|I|$. But $I \setminus (J_t \cap I)$ is an independent set that is contained within G_t, and thus our algorithm returned an independent set no smaller than this one. Note that there is no particular reason that the maximum independent set of G_i should be the restriction of a global maximum independent set, but it is certainly no smaller.

For the pedantic, we note that we have overestimated the time complexity: we should actually have $O(2^{3(k-1)}n) = O(8^{k-1}n)$ time for finding a maximum independent set in each G_i. Furthermore, that n is in fact the number of vertices in G_i, not in G, and so is probably somewhat smaller then n, and in fact at most $n-3$. Of course these observations do not fundamentally affect the result. $\square$

Essentially, we created a family of subgraphs $G_i = G[V \backslash J_i]$, $1 \leq i \leq k$, where the J_i partition the vertex set. By choosing the subgraphs in a certain way, we obtain a special case (k-outerplanarity) where we can easily compute a maximal independent set. Since the J_i form a partition, simple averaging guarantees that the maximum independent set among the G_i is at least $\frac{k-1}{k}$ of the size of a maximum independent set in the original graph.

4 Independent sets and colouring

We mention a somewhat related research problem. Recall the problem of vertex colouring. This can be viewed as partitioning the graph into independent sets, with the goal of having as few sets as possible. It is known that every planar graph is 4-colourable, so applying the previous simple observation, there is at least one independent set of size at least $\frac{1}{4}n$ in any planar graph.

In fact a 4-colouring (and hence an independent set of size at least $\frac{1}{4}n$) can be found for planar graphs in $O(n^2)$ time, with a large constant. Using Baker's algorithm, we can find an independent set of size at least $(\frac{1}{4} - \epsilon)n$ in $O(n)$ time, although here the constant is exponential in $\frac{1}{\epsilon}$. A greedy algorithm will give an independent set of size at least $\frac{1}{5}n$ in $O(n)$ time. In fact, one can find a 5-colouring of a planar graph in $O(n)$ time [CNS81], and simply take the largest colour class. Biedl has an algorithm that can give an independent set of size at least $\frac{5}{23}n$ in $O(n)$ time [Bie99].

The question is can we exploit the fact that we know there is such a set to efficiently produce an independent set of size at least $\frac{1}{4}n$ in $O(n)$ time. This is not known, and it is proposed that this is a hard problem.

5 Approximating minimal weighted $\frac{2}{3}$-separators in a planar graph

Now we turn ti another approximation algorithm for independent sets, for which we need separators. Recall that a weighted $\frac{2}{3}$ separator of a graph is a set S of vertices such that every component of $G[V \setminus S]$ has weight at most $\frac{2}{3}w(V)$. The following result is due to Lipton and Tarjan [LT79], which we give without proof.

Theorem 4 *Every planar graph has a weighted $\frac{2}{3}$ separator of size at most $2\sqrt{2}n$.* $\square$

This is, up to a constant factor, best possible. Consider the grid graph, and more specifically, the square grid graph (so it is a $\sqrt{n} \times \sqrt{n}$ grid). We showed in an assignment that it has a weighted $\frac{2}{3}$ separator of size $\sqrt{n}$. With a separator any smaller, we cannot cut across the entire grid. In other words, there must be a column and a row that do not intersect the separator. We omit a formal proof, but this means that any separator of size less than $\sqrt{n}$ will leave a component larger than $\frac{2}{3}$ of the total, even for a uniform weight function.

Lipton and Tarjan also proved the following generalization [LT80].

Theorem 5 *Let G be a weighted planar graph, and $0 < \epsilon < 1$. There is a weighted ϵ-separator of size at most $16\sqrt{\frac{n}{\epsilon}}$. Such a separator can be found in $O(n \log n)$ time.* $\square$

We give a brief sketch of the proof; for more details see [LT80]. The basic idea of the proof is a divide-and-conquer strategy. We first take an initial separator of the graph, and then recursively find separators for the components that are "too big" (i.e., components of size larger than ϵn). By construction, the union of all of the separators will be an ϵ separator. It can be shown that the total size is no greater than $16\sqrt{\frac{n}{\epsilon}}$.

The constant in Theorem 4 can in fact be improved to $\sqrt{6}$, and in Theorem 5 to 4 [Bod98].

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