Chapter 6

Lecture 6

Advanced Topics in Graph Algorithms
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6.1 Some Optimization Algorithms on Triangulated Graphs

6.1.1 Computing the chromatic number and all maximal cliques

Theorem 6.1 (Fulkerson and Gross 65) *Let $G = (V, E)$ be a triangulated graph, and let σ be a perfect elimination order for G. Every maximal clique in G is of the form: $X_u \cup \{u\}$ where $X_u = \{x \in Adj(u) | \sigma^{-1}(u) < \sigma^{-1}(x)\}$.*

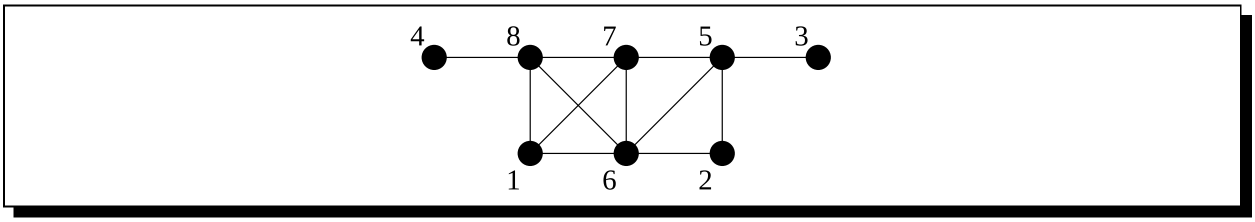

Figure 6.1: An example for maximal cliques in a graph

Example: In the chordal graph in Figure 6.1, vertices are numbered according to p.e.o and we have: $X_1 \cup \{1\} = \{1, 6, 7, 8\}$, $X_2 \cup \{2\} = \{2, 5, 6\}, X_3 \cup \{3\} = \{3, 5\}, X_4 \cup \{4\} =$

$\{4,8\}, X_5 \cup \{5\} = \{5,6,7\}$ - maximal cliques, and: $X_6 \cup \{6\} = \{6,7,8\}, X_7 \cup \{7\} = \{7,8\}$ - non maximal cliques.

Proof: Let C be an arbitrary maximal clique in G, and let v be the first vertex according to the order σ which is contained in C. Let $w \in G, \sigma^{-1}(w) > \sigma^{-1}(v)$. If $w \notin X_v \cup \{v\}$, then $wv \notin E$, therefore $w \notin C$, so we get: $C \subseteq X_v \cup \{v\}$.

On the other hand, since G is triangulated $X_v \cup \{v\}$ is a clique, and because C is maximal we get: $C = X_v \cup \{v\}$. ∎

Corollary 6.2 *A triangulated graph on n vertices has at most n maximal cliques, with equality if and only if the graph has no edges.*

Proof: Every maximal clique is characterized uniquely by the first vertex in σ which is contained in it. ∎

From Theorem 6.1, a naive algorithm for calculating the maximum clique in a triangulated graph is as follows:

- Find $\{v\} \cup X_v$ for each v.
- Filter out the non-maximal cliques.

A more efficient algorithm is based on the algorithm TEST for testing whether a given vertices permutation σ is a PEO as shown in lecture no. 4. In that algorithm X_v is calculated for each v.

Theorem 6.3 *When algorithm TEST operates on a chordal graph and a p.e.o. σ, $\{u\} \cup X_u$ is not a maximal clique iff X_u is concatenated to $A(u)$.*

Proof: Reminder: In the phase of eliminating v (phase $\sigma^{-1}(v) = i$ of TEST), $X_v - \{u\}$ is concatenated to A(u) if u is the first neighbor of v after it in σ.

($\Longrightarrow$) If $X_u \cup \{u\}$ is not a maximal clique then there exists a vertex w, $\sigma^{-1}(w) < \sigma^{-1}(u)$ such that $\{u\} \cup X_u \subset \{w\} \cup X_w$. Let us take v such that $v = \sigma(\max\{\sigma^{-1}(w) | \sigma^{-1}(w) < \sigma^{-1}(u)\})$.

In phase $\sigma^{-1}(v)$, $X_u = X_v - \{u\}$ is concatenated to A(u).

($\Longleftarrow$) Assume that at phase $\sigma^{-1}(v)$ $X_u = X_v - \{u\}$ is concatenated to A(u). Then $X_v = X_u \cup \{u\}$. So $X_v \cup \{v\}$ is a clique and $X_u \cup \{u\} \subset X_v \cup \{v\}$, therefore $X_u \cup \{u\}$ is not maximal. ∎

See the modified algorithm in figure 6.2.

Theorem 6.4 *The above algorithm correctly calculates the chromatic number and all maximal cliques of a triangulated graph $G = (V, E)$ in $O(|V| + |E|)$ time.*

The proof is similar to that of the algorithm in lecture no. 4.

1. $\chi \leftarrow 1$
2. for all vertices v do $S(v) \leftarrow 0$
3. for $i \leftarrow 1$ to n do
    begin
4. $v \leftarrow \sigma(i)$.
5. $X \leftarrow \{x \in \mathrm{Adj}(v) | \sigma^{-1}(v) < \sigma^{-1}(x)\}$
6.   if $\mathrm{Adj}(v) = \emptyset$ then print $\{v\}$.
7.   if $X = \emptyset$ then goto 13
8. $u \leftarrow \sigma(\min\{\sigma^{-1}(x) | x \in X\})$
9. $S(u) \leftarrow \max\{S(u), |X| - 1\}$
10.   if $S(v) < |X|$ then do
        begin
11.   print $\{v\} \cup X$
12. $\chi = \max\{\chi, 1 + |X|\}$
        end
13. end
14. print "The chromatic number is ",χ.

Figure 6.2: Chromatic Number Calculation

6.1.2 Finding α and k

Next we tackle the problem of finding the stability number $\alpha(G)$ of a triangulated graph G. Better yet, since G is perfect, the clique cover size $k(G) = \alpha(G)$.

Let σ be a perfect elimination scheme for $G = (V, E)$. We define inductively a sequence of vertices $y_1, y_2, \ldots, y_t$ in the following manner:

(1) $y_1 = \sigma(1)$

(2) $y_i = \sigma(\min\{\sigma^{-1}(x) | \sigma^{-1}(x) > \sigma^{-1}(y_{i-1}) \text{ and } x \notin Xy_1 \cup Xy_2 \cup \ldots \cup Xy_{i-1}\})$

y_i is the first vertex in σ which appears after y_{i-1} in σ, and which is not in $Xy_1 \cup Xy_2 \cup \cdots \cup Xy_{i-1}$. All vertices following y_t are in $Xy_1 \cup Xy_2 \cup \cdots \cup Xy_t$.

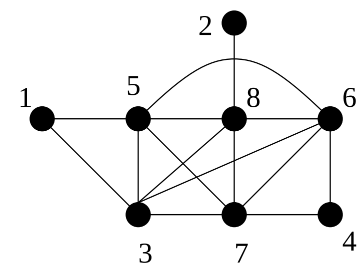

Figure 6.3: An example for finding maximal stable set

For example, in the graph in Figure 6.3:

$y_1 = 1,\ Xy_1 = \{3, 5\}$

$y_2 = 2,\ Xy_2 = \{8\}$

$y_3 = 4,\ Xy_3 = \{6, 7\}$

Here $\{y_1, y_2, y_3\}$ is a maximum stable set, and:

$y_1 \cup Xy_1 = \{1, 3, 5\},\ y_2 \cup Xy_2 = \{2, 8\},\ y_3 \cup Xy_3 = \{4, 6, 7\}$ are a minimum clique cover.

Theorem 6.5 *(Gavril, 1972) The set $\{y_1, y_2, \ldots, y_t\}$ is a maximum stable set of G, and the collection of sets $Y_i = \{y_i\} \cup Xy_i,\ i = 1, 2, \ldots, t$ comprises a minimum clique cover of G.*

Proof: The set $y_1, y_2, \ldots, y_t$ is stable since if $y_i y_j \in E$ for $i < j$, then $y_j \in Xy_i$, which cannot be. Thus $\alpha(G) \geq t$.

On the other hand, since $\{Y_1, Y_2, \ldots, Y_t\}$ is a set of cliques which together cover G, we get $k(G) \leq t$.

Since $\alpha(G) \leq k(G)$ we get: $\alpha(G) = k(G) = t$, and we have produced the desired maximum stable set and minimum clique cover. ∎

The method above, with a slight modification in the algorithm of section 6.1.1, finds a maximum stable set and a minimum clique cover in time complexity of $O(|V| + |E|)$.

6.2 Comparability Graphs

6.2.1 Some Definitions

An undirected graph $G = (V, E)$ is a *comparability graph* if there exists an orientation (V, F) of G satisfying

$F \cap F^{-1} = \emptyset$,

$F + F^{-1} = E$,

$F^2 \subseteq F$, where $F^2 = \{ac | ab, bc \in F$ for some vertex $b\}$

F is also called a *transitive orientation* of G, and G is also called *transitively orientable* (TRO).

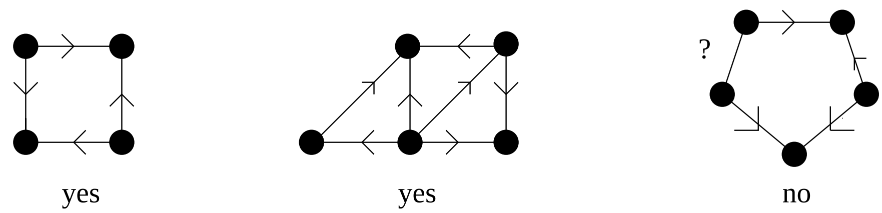

Figure 6.4: some comparability graphs

The relation F is a strict partial ordering (irreflexive, transitive). Comparability graphs are also known as *partially orderable* graphs.

Define the binary relation Γ on the edges of an undirected graph $G = (V, E)$ as follows: $ab\Gamma a'b' \iff a = a'$ and $bb' \notin E$ or $b = b'$ and $aa' \notin E$.

We say that ab *directly forces* $a'b'$ whenever $ab\Gamma a'b'$.

The reflexive, transitive closure of Γ is Γ^*. It is easily shown that Γ^* is an equivalence relation on E, and hence partitions E into what we shall call the *implication classes* of G.

Thus edges ab and cd are in the same implication class iff there exists a sequence of edges $ab = a_0b_0\Gamma a_1b_1\Gamma \ldots \Gamma a_kb_k = cd$, $k \geq 0$. Such a sequence is called a Γ-*chain* from ab to cd, and we say that ab (eventually) *forces* cd whenever $ab\Gamma^*cd$.

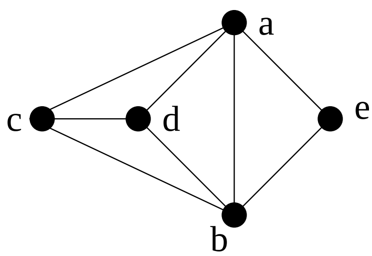

Figure 6.5: An example for implication classes

For example, the 8 implication classes in the graph of figure 6.5 are: $A_1 = \{ab\}$, $A_2 = \{cd\}$, $A_3 = \{ac, ad, ae\}$, $A_4 = \{bc, bd, be\}$, $A_1^{-1} = \{ba\}$, $A_2^{-1} = \{dc\}$, $A_3^{-1} = \{ca, da, ea\}$, $A_4^{-1} = \{cb, db, eb\}$.

Notice that $ba\Gamma dc \iff ab\Gamma cd$ and $ba\Gamma^* dc \iff ab\Gamma^* cd$, by definition. Notice also that $ab\Gamma ab$ (by our convention that G contains no self-loops), but *not* $ab\Gamma ba$ necessarily.

Let $I(G)$ denote the collection of implication classes of G. We define $\hat{I}(G) = \{\hat{A} | A \in I(G)\}$, where $\hat{A} = A \cup A^{-1}$ (the symmetric closure of $\hat{A}$). The members of $\hat{I}(G)$ are called the *color classes* of G.

6.2.2 Implication Classes

Theorem 6.6 *Let A be an implication class of an undirected graph G. If G has a transitive orientation F, then either $F \cap \hat{A} = A$ or $F \cap \hat{A} = A^{-1}$ and, in either case, $A \cap \hat{A} = \emptyset$.*

Proof: We defined Γ in order to capture the fact that, for any transitive orientation F of G, if $ab\Gamma a'b'$ and $ab \in F$, then $a'b' \in F$. Applying this property repeatedly, we obtain $F \cap A = \emptyset$ or $A \subseteq F$.

If $F \cap A = \emptyset \Rightarrow A \subseteq F^{-1} \Rightarrow A^{-1} \subseteq F \Rightarrow F \cap \hat{A} = A^{-1}$

If $A \subseteq F \Rightarrow A^{-1} \subseteq F^{-1} \Rightarrow F \cap A^{-1} = \emptyset$ (since $F \cap F^{-1} = \emptyset$) $\Rightarrow F \cap \hat{A} = A$.

In both cases $A \cap A^{-1} = \emptyset$ (since $F \cap F^{-1} = \emptyset$). ■

We shall prove later that the converse of Theorem 6.6 is also valid, namely if $A \cap A^{-1} = \emptyset$ for every implication class A, then G has a transitive orientation.

Note that the existence of an arbitrary union of implication classes $F = \cup_i A_i$ satisfying $F \cap F^{-1} = \emptyset$ and $F + F^{-1} = E$ does not necessarily mean that F is a transitive orientation of G. For example, in a simple triangle abc, $\{ab\}$, $\{bc\}$ and $\{ca\}$ are implication classes satisfying the condition but the resulting orientation is cyclic.

Lemma 6.7 *If $ab\Gamma^* cd$, then there exists a Γ-chain from ab to cd of the form*

$$ab = a_0b_0\Gamma a_1b_0\Gamma a_1b_1\Gamma a_2b_1 \ldots \Gamma a_kb_k = cd$$

Proof: In the Γ-chain in the lemma the indices change for the a_i-s in the odd "Γ"-s, and for the b_i-s in the even ones. Let us look at any Γ-chain between ab and cd. If in that chain there is a group of 3 or more sequential edges with the same index changed - in particular in that group the first and last edges directly force each other. We shall take for "our" chain those first and last edges - so we got only 2 edges in a row with the same index changed. We shall do that for every such group of 3 or more edges in a row, and will get the desired Γ-chain. ■

A Γ-chain satisfying the lemma is called a *canonical Γ-chain*.

6.2.3 The Triangle Lemma

Lemma 6.8 (The Triangle Lemma) *Let A, B and C be implication classes of an undirected graph $G = (V, E)$ with $A \neq B$ and $A \neq C^{-1}$ and having edges $ab \in C$, $ac \in B$ and $bc \in A$*

- *(i) If $b'c' \in A$ then $ab' \in C$ and $ac' \in B$*
- *(ii) If $b'c' \in A$ and $a'b' \in C$, then $a'c' \in B$*
- *(iii) No edge in A touches the vertex a*

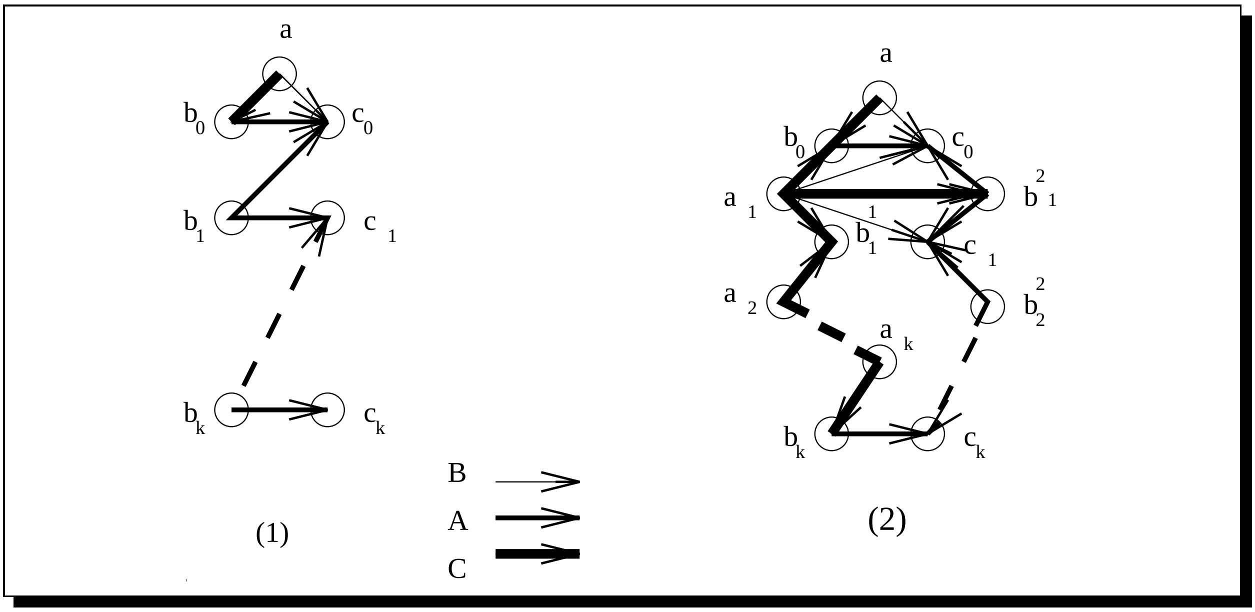

Figure 6.6: The triangle lemma

Proof: If $b'c' \in A$, by Lemma 6.7 there exists a canonical Γ-chain

$$bc = b_0c_0\Gamma b_1c_0\Gamma b_1c_1 \ldots \Gamma b_kc_k = b'c'$$

(see Figure 6.6(1)). Let $a = a_0$.

$a_0c_0 \in B$ doesn't force $b_1c_0 \in A \Rightarrow a_0b_1 \in E$.

$a_0b_0 \in C, b_0b_1 \notin E$ (since $b_0c_0\Gamma b_1c_0$) $\Rightarrow a_0b_1 \in C$.

$a_0b_1 \in C, b_1c_1 \in A \Rightarrow$ $a_0c_1 \in E$ (otherwise $b_1a_0\Gamma b_1c_1$, contradiction to $C^{-1} \neq A$).

$c_0c_1 \notin E, a_0c_0 \in B \Rightarrow a_0c_1 \in B$. By induction on the same argument we get: $a_0b_k \in C, a_0c_k \in B$. This proves (i). In particular (i)$\Rightarrow$(iii).

Next, assume $b'c' \in A, a'b' \in C$. Consider a new canonical Γ-chain

$$ab = a_0b_0\Gamma a_1b_0\Gamma a_1b_1\Gamma \ldots a_lb_l = a'b'$$

(see Figure 6.6(2)). If $ca_1 \notin E$ then $ba_1\Gamma bc$ but $C^{-1} \neq A$ therefore $ca_1 \in E$.

$aa_1 \notin E, ac \in B \Rightarrow a_1c \in B$.

The triangle a_1bc has the same characteristics as the original triangle abc. By induction we get that the triangle $a_lbc = a'bc$ has also the same characteristics. From (i) (with: a' instead of a) we get $a'c' \in B$. ■

Theorem 6.9 *Let A be an implication class of an undirected graph $G = (V, E)$. Exactly one of the following alternatives holds:*

- *(i) $A = \hat{A} = A^{-1}$*
- *(ii) $A \cap A^{-1} = \emptyset, A$ and A^{-1} are transitive orientations of $\hat{A}$.*

Proof: (i) Assume $A \cap A^{-1} \neq \emptyset$. Let $ab \in A \cap A^{-1}$. So $ab\Gamma^*ba$. According to the definition of Γ^* for any $cd \in A, cd\Gamma^*ab$ iff $dc\Gamma^*ba$. Since Γ^* is an equivalence relation and $ab\Gamma^*ba$, $cd\Gamma^*dc$ and therefore $dc \in A$. Thus $A = \hat{A}$.

(ii) Assume $A \cap A^{-1} = \emptyset$ and let $ab, bc \in A$. We shall now prove $ac \in A$. Now $ac \notin E \Rightarrow ab\Gamma cb \Rightarrow cb \in A \Rightarrow bc \in A^{-1}$, a contradiction. Thus $ac \in E$. Let B be the implication class of G containing ac, and suppose $A \neq B$.

Since $A \neq A^{-1}$ and $ab \in A$, the triangle lemma (iii) on triangle abc (with $bc \in A, ac \in B, ab \in C = A$) gives us that there is no edge in A touching a.

But $ab \in C = A$ touches a. A contradiction.

Therefore $ac \in A \Rightarrow A$ is transitive $\Rightarrow A^{-1}$ is transitive.

Finally, A is an implication class of $\hat{A}$, so by Theorem 6.6 A and A^{-1} are the only transitive orientations of $\hat{A}$. ■

Corollary 6.10 *Each color class of an undirected graph G either has exactly two transitive orientations, one being the reversal of the other, or has no transitive orientation. If in G there is a color class having no transitive orientation, then G is not a comparability graph.*

6.2.4 Decomposition of Graphs

Let H_0 be a graph with n vertices $v_1, v_2, \ldots, v_n$ and let $H_1, H_2, \ldots, H_n$ be n disjoint graphs. The *composition graph* $H = H_0[H_1, H_2, \ldots, H_n]$ is formed as follows:

For all $1 \leq i, j \leq n$, replace vertex v_i in H_0 with the graph H_i and make each vertex H_i adjacent to each vertex of H_j whenever v_i is adjacent to v_j in H_0.

Formally, for $H_i = (V_i, E_i)$ we define $H = (V, E)$ as follows:

$$V = \cup_{i \geq 1} V_i$$

$E = \cup_{i \geq 1} E_i \cup \{xy | x \in V_i,\ y \in V_j \text{ and } v_i v_j \in E_0\}$

We may also denote $E = E_0[E_1, E_2, \ldots, E_n]$

H_0 is called the *outer factor* of H.

$H_i, i = 1, \ldots, n$ are called the *inner factors* of H.

Theorem 6.11 *Let $G = G_0[G_1, G_2, \ldots, G_n]$ where the G_i-s are disjoint undirected graphs. Then G is a comparability graph iff each G_i $(0 \leq i \leq n)$ is a comparability graph.*

Proof:

($\Longrightarrow$) It is obvious that if one of the G_i-s is not a TRO graph then G is not TRO.

($\Longleftarrow$) Let $F_0, F_1, \ldots, F_n$ be transitive orientations of $G_0, G_1, \ldots, G_n$, respectively. Let us show that $F = F_0[F_1, F_2, \ldots, F_n]$ is transitive: Let $ab, bc \in F$: if $ab, bc \in F_i$ then $ac \in F_i$ by transitivity. If $ab \in F_i,\ bc \in F_j,\ j \neq i$ then we cannot have $i \neq 0,\ j \neq 0$ since inner factors are disjoint. If $ab \in F_i$ $bc \in F_0$ then $a \in G_i$ so, since every vertex in G_i is connected to every vertices in the factor in which c is contained in, then $ac \in F_0$. If $ab \in F_0$ $bc \in F_i$ then $c \in G_i$ so $ac \in F_0$,]

and in both cases we get transitivity. ∎

A graph is called *decomposable* if it can be expressed as a non-trivial composition of some of its induced subgraphs.

(For each graph there is the trivial composition: $G = k_1[G]$.)

Formally, $G = (V, E)$ is decomposable if there exists a partition $V = V_1 + V_2 + \cdots + V_r$ of the vertices into nonempty pairwise disjoint subsets with $1 < r < |V|$ such that $G = G_R[G_{V1}, G_{V2}, \ldots, G_{Vr}]$ for any set of representatives $R = \{x_1, x_2, \ldots, x_r\}, x_i \in V_i$.

Such a partition is said to *induce a proper decomposition of* G.