CMSC 313 Lecture 16

- Postulates & Theorems of Boolean Algebra
- Semiconductors
- CMOS Logic Gates

Last Time

- Overview of second half of this course
- Logic gates & symbols
- Equivalence of Boolean functions, truth tables, Boolean formulas and combinational circuits
- Universality of NAND gates

Postulates of Boolean Algebra

- Commutative: $AB = BA, \quad A + B = B + A$

- Associative: $(AB)C = A(BC), \quad (A + B) + C = A + (B + C)$

- Distributive: $A(B + C) = AB + AC, \quad A + BC = (A + B)(A + C)$

- Identity: there exists 0 and 1 such that for all A,

$$1A = A \quad \text{and} \quad 0 + A = A.$$

- Complement: for all A, there exists $\overline{A}$ such that

$$A\overline{A} = 0 \quad \text{and} \quad A + \overline{A} = 1$$

where 0 and 1 are the identity elements.

Some Theorems of Boolean Algebra

- Zero and One: $0A = 0, \quad 1 + A = 1$
- Idempotence: $AA = A, \quad A + A = A$
- Involution: $\overline{\overline{A}} = A$
- DeMorgan's: $\overline{AB} = \overline{A} + \overline{B}, \quad \overline{A + B} = \overline{A}\,\overline{B}$
- Absorption: $A(A + B) = A, \quad A + AB = A$
- Consensus: $AB + \overline{A}C + BC = AB + \overline{A}C,$
  $(A + B)(\overline{A} + C)(B + C) = (A + B)(\overline{A} + C)$

Theorems are derived from the postulates

- Idempotence:

$$
\begin{aligned}
A &= A1 && \text{identity, commutative} \\
&= A(A + \overline{A}) && \text{complement} \\
&= AA + A\overline{A} && \text{distributive} \\
&= AA + 0 && \text{complement} \\
&= AA && \text{commutative, identity}
\end{aligned}
$$

$$
\begin{aligned}
A &= 0 + A && \text{identity} \\
&= (A\overline{A}) + A && \text{complement} \\
&= (A + A)(\overline{A} + A) && \text{distributive} \\
&= (A + A)1 && \text{commutative, complement} \\
&= A + A && \text{commutative, identity}
\end{aligned}
$$

Theorems are derived from the postulates

- Zero and One:

$$
\begin{aligned}
0A &= (A\overline{A})A && \text{complement} \\
&= A(A\overline{A}) && \text{commutative} \\
&= (AA)\overline{A} && \text{associative} \\
&= A\overline{A} && \text{idempotent} \\
&= 0 && \text{complement}
\end{aligned}
$$

$$
\begin{aligned}
1 + A &= (A + \overline{A}) + A && \text{complement} \\
&= A + (A + \overline{A}) && \text{commutative} \\
&= (A + A) + \overline{A} && \text{associative} \\
&= A + \overline{A} && \text{idempotent} \\
&= 1 && \text{complement}
\end{aligned}
$$

Theorems are derived from the postulates

- Absorption:

$$\begin{aligned} A + AB &= A1 + AB && \text{identity, commutative} \\ &= A(1+B) && \text{distributive} \\ &= A1 && \text{one} \\ &= A && \text{commutative, identity} \end{aligned}$$

$$\begin{aligned} A(A+B) &= AA + AB && \text{distributive} \\ &= A + AB && \text{idempotent} \\ &= A && \text{absorption} \end{aligned}$$

Proof by truth table?

- Absorption: $A + AB = A$

A	B	AB	$A + AB$
0	0	0	0
0	1	0	0
1	0	0	1
1	1	1	1

- Proof by truth table only applies to 0-1 Boolean Algebra.

- Proof by derivation from postulates holds for any Boolean Algebra.

A Boolean Algebra with 8 elements

- Elements are the subsets of $\{a, b, c\}$:

$$\emptyset,\ \{a\},\ \{b\},\ \{c\},\ \{a,b\},\ \{a,c\},\ \{b,c\},\ \{a,b,c\}$$

- Operations: $AB \rightarrow A \cap B,\ \ A + B \rightarrow A \cup B$.
- Union and intersection are commutative and associative.
- Union distributes over intersection: $A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$.
- Intersection distributes over union: $A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$.
- Identity: $0 = \emptyset,\ \ 1 = \{a, b, c\},\ \ \{a, b, c\} \cap A = A$ and $\emptyset \cup A = A$.
- Complement: $\overline{A} = \{a, b, c\} - A,\ \ A \cap \overline{A} = \emptyset$ and $A \cup \overline{A} = \{a, b, c\}$.
- All postulates hold. Therefore, all derived theorems also hold.

Simplifying MAJ3

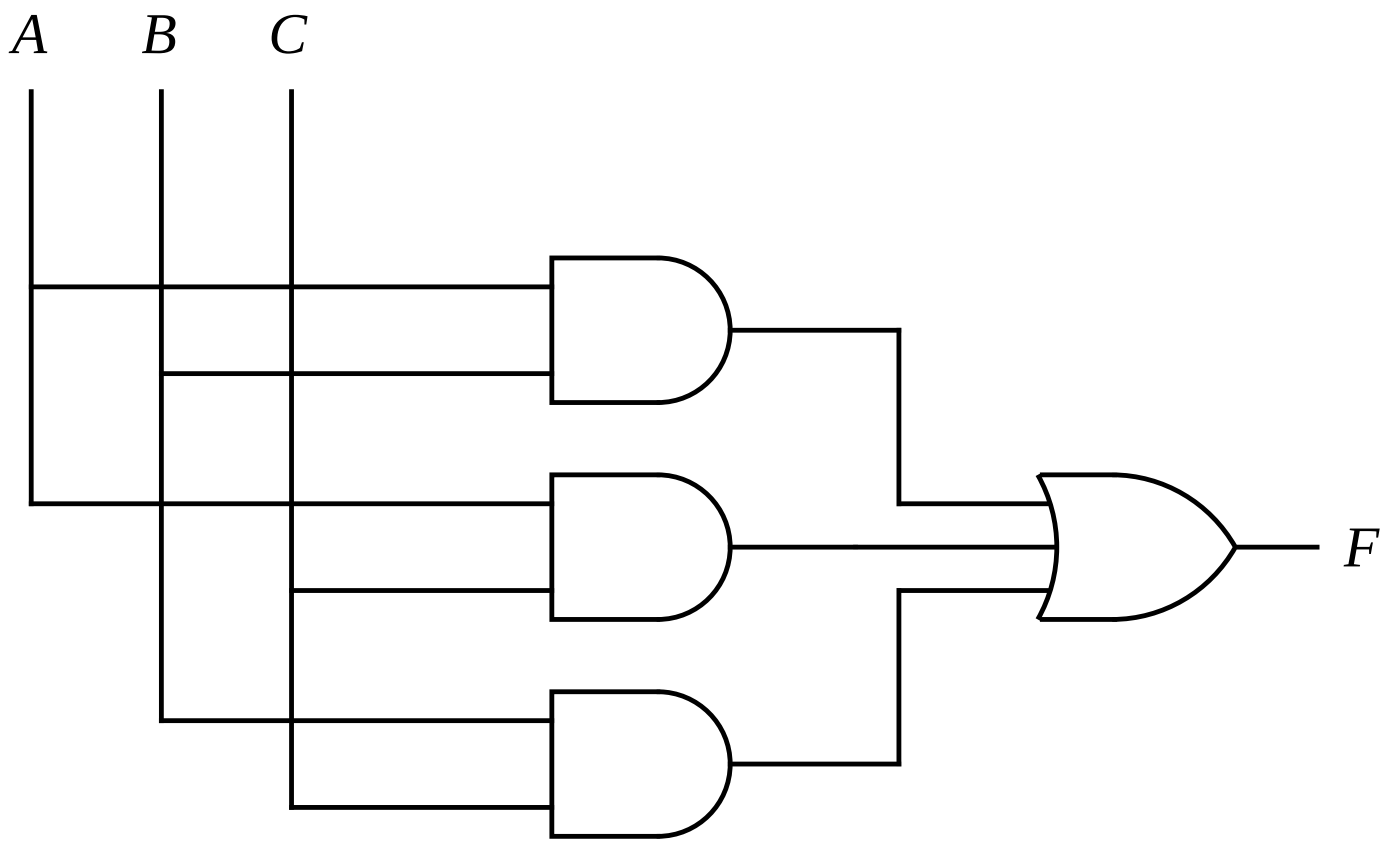
- Simplify MAJ3 using the postulates and theorems of Boolean Algebra

$$\begin{aligned}
&\mathrm{MAJ3}(A, B, C) \\
&= \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC && \text{SOP form} \\
&= \overline{A}BC + A\overline{B}C + AB\overline{C} + ABC + ABC + ABC && \text{idempotent} \\
&= \overline{A}BC + ABC + A\overline{B}C + ABC + AB\overline{C} + ABC && \text{commutative} \\
&= (\overline{A} + A)BC + (\overline{B} + B)AC + (\overline{C} + C)AB && \text{distributive} \\
&= 1BC + 1AC + 1AB && \text{complement} \\
&= BC + AC + AB && \text{identity}
\end{aligned}$$

- Resulting circuit uses fewer gates.

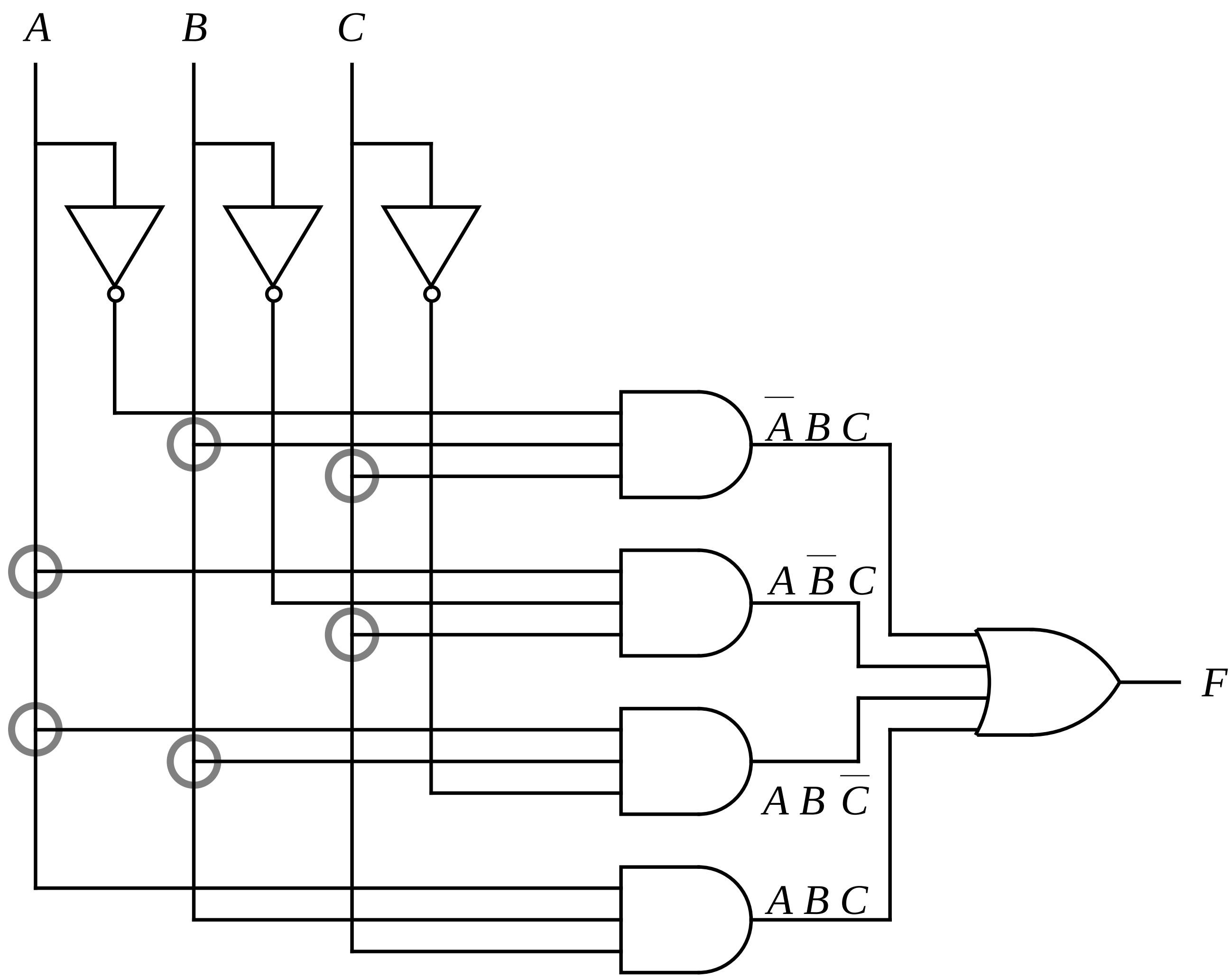
The Algebraic Method

- This majority circuit is functionally equivalent to the previous majority circuit, but this one is in its minimal two-level form:



AND-OR Implementation of Majority

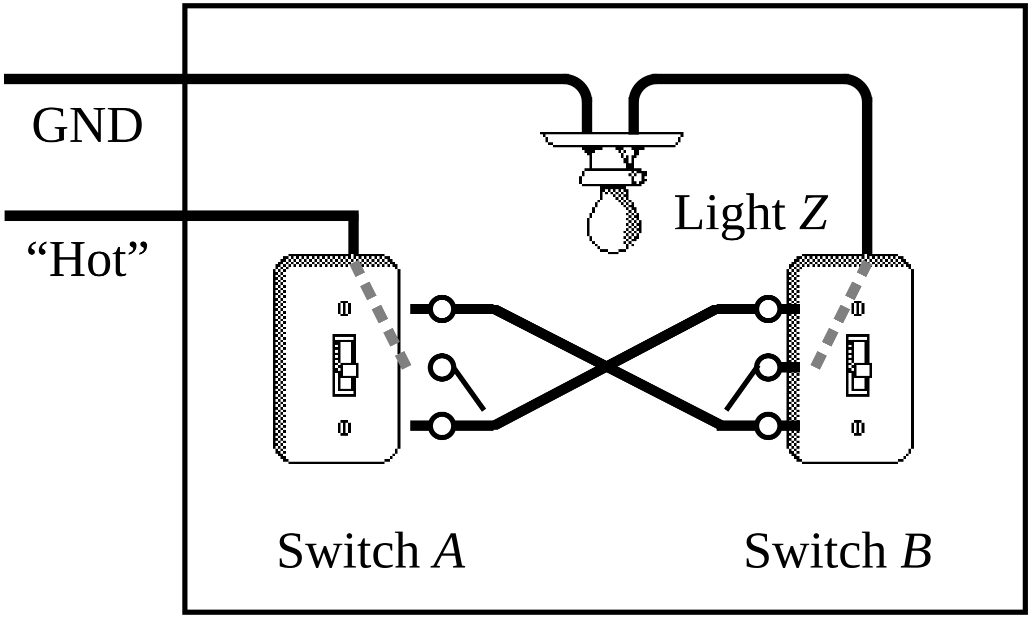
- Gate count is 8, gate input count is 19.



How do we make gates???

A Truth Table

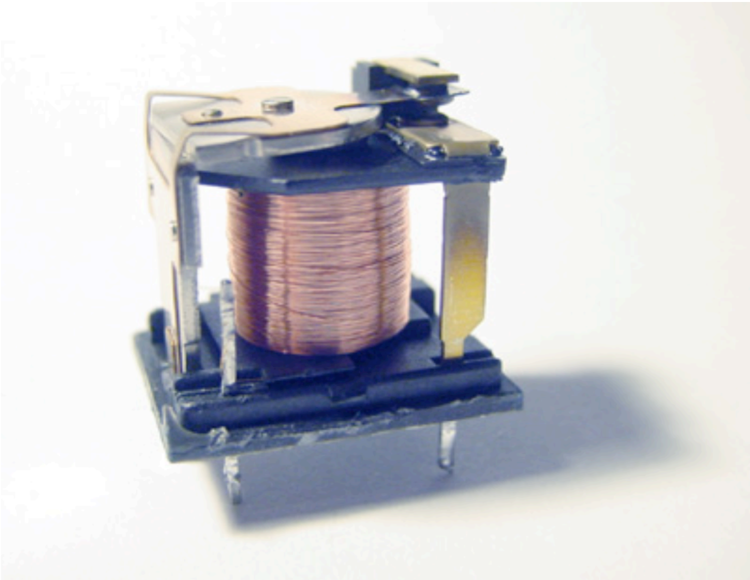
- Developed in 1854 by George Boole.
- Further developed by Claude Shannon (Bell Labs).
- Outputs are computed for all possible input combinations (how many input combinations are there?)
- Consider a room with two light switches. How must they work?

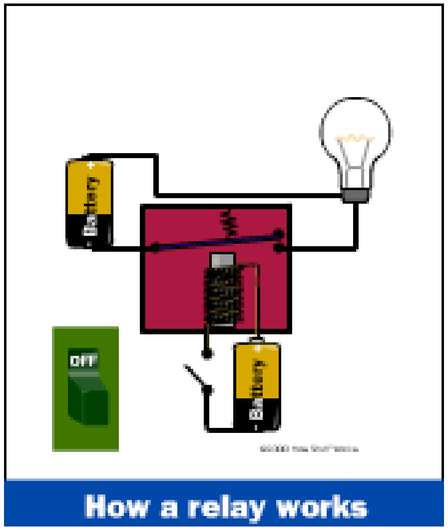


Inputs		Output
A	*B*	*Z*
0	0	0
0	1	1
1	0	1
1	1	0

Electrically Operated Switch

- Example: a relay





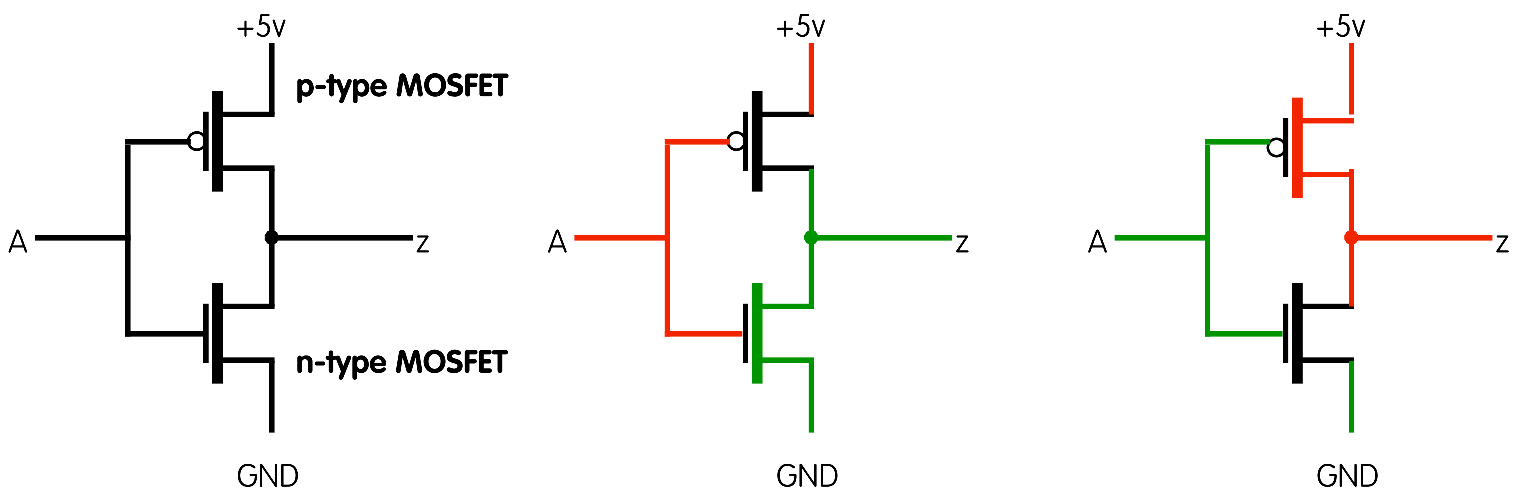
source: http://www.howstuffworks.com/relay.htm

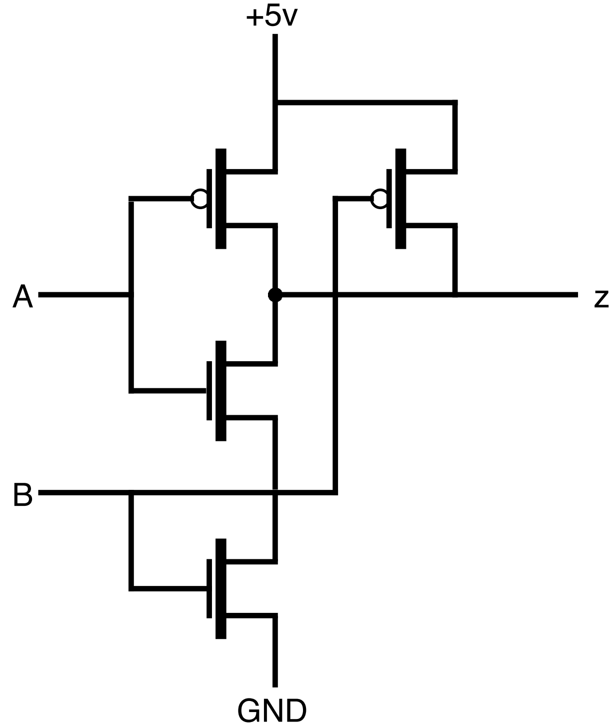
Semiconductors

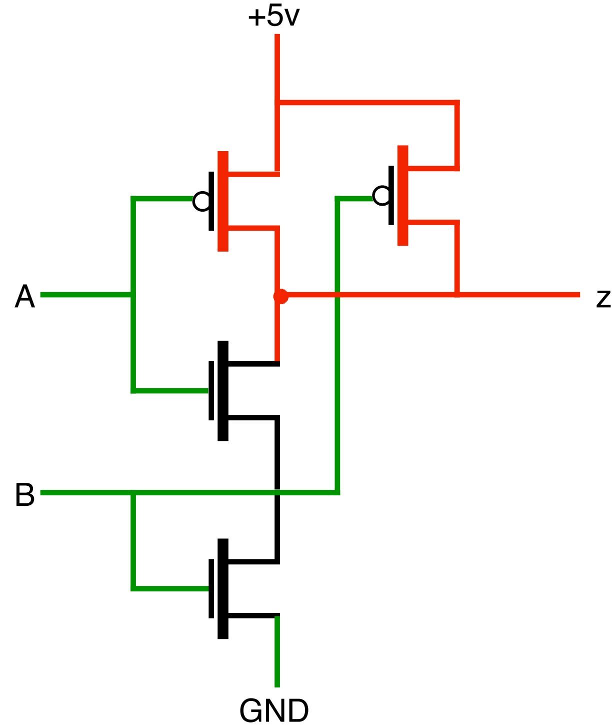
- Electrical properties of silicon
- Doping: adding impurities to silicon
- Diodes and the P-N junction
- Field-effect transistors

An Inverter using MOSFET

- **CMOS = complementary metal oxide semiconductor**
- **P-type transistor conducts when gate is low**
- **N-type transistor conducts when gate is high**

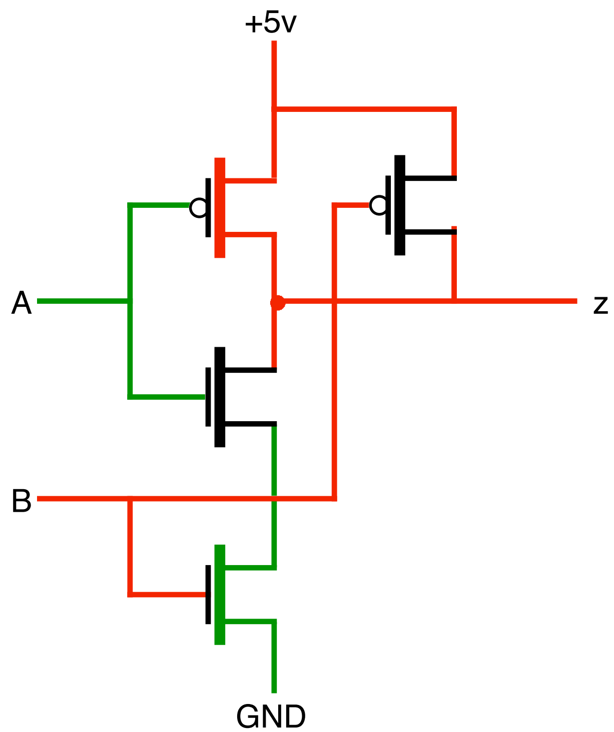


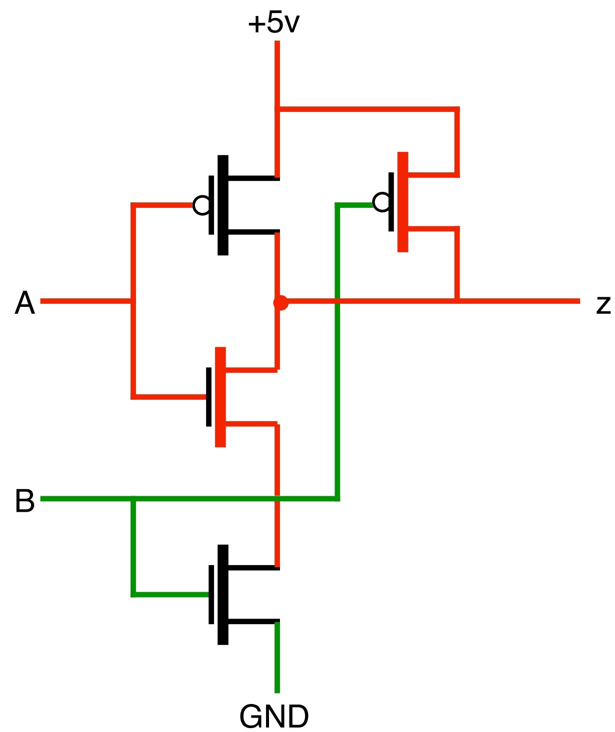


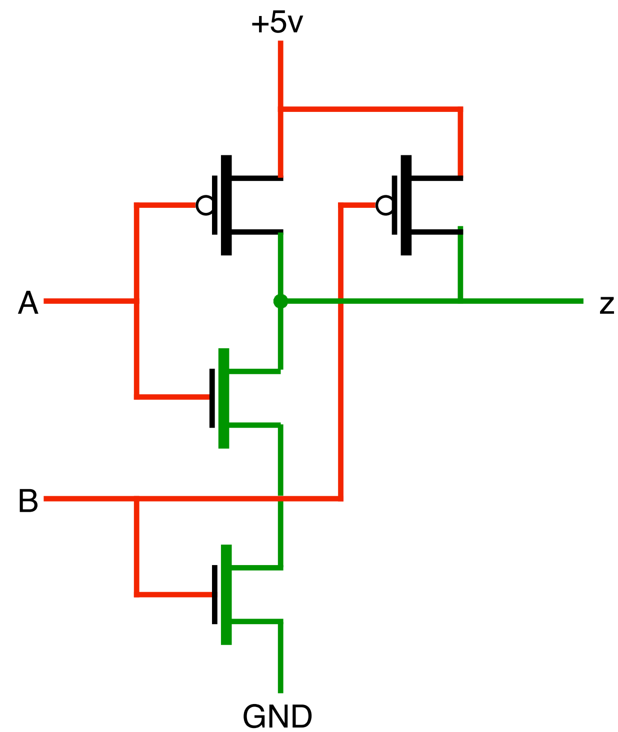


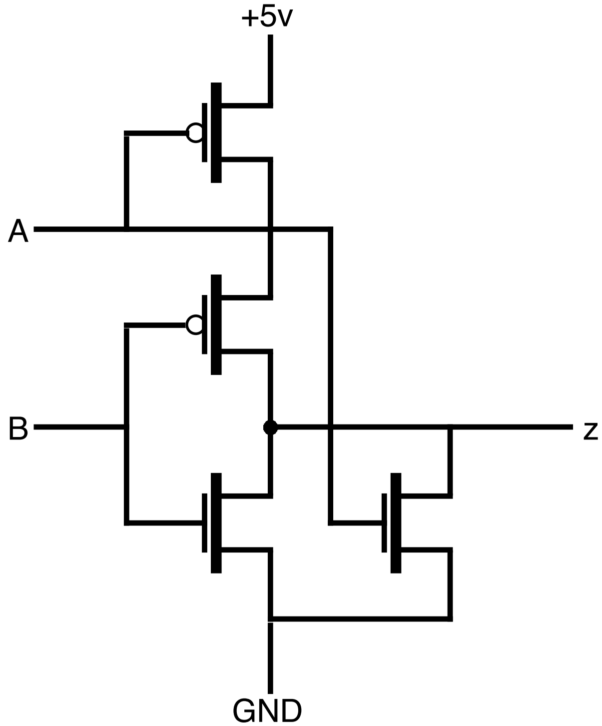
NAND GATE

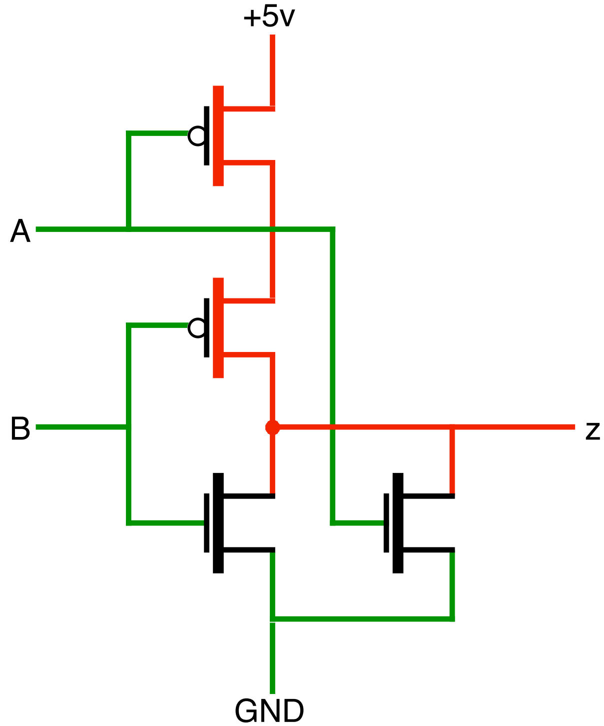
A	B	z
0	0	1
0	1	1
1	0	1
1	1	0





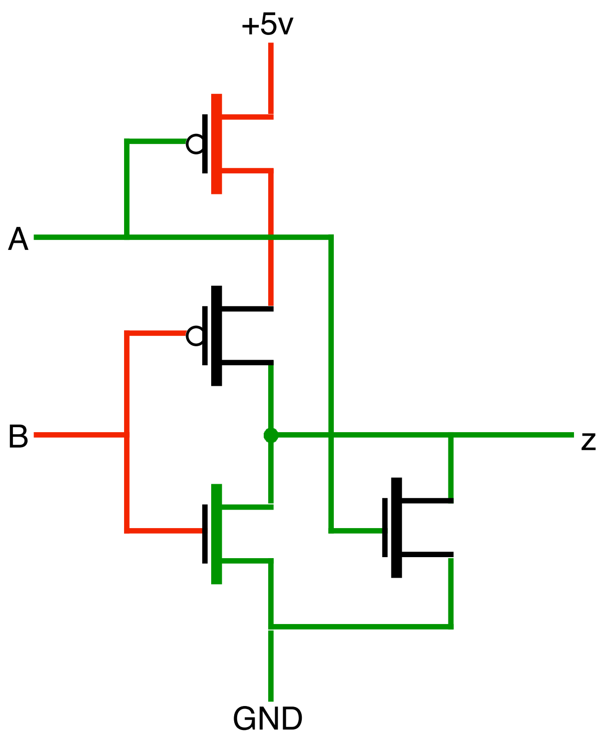


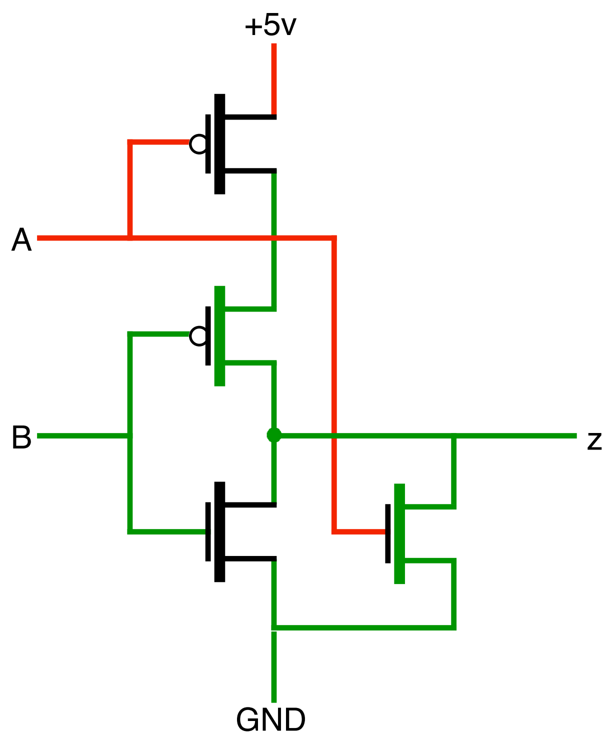


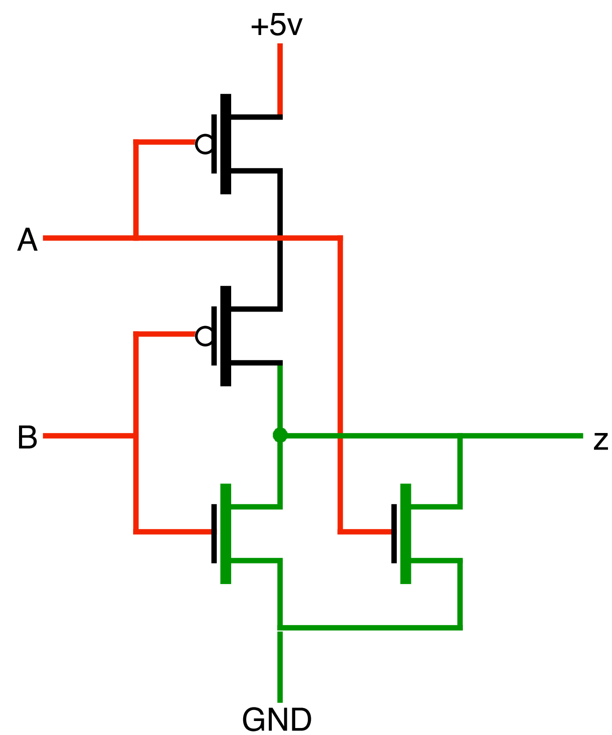


NOR GATE

A	B	z
0	0	1
0	1	0
1	0	0
1	1	0





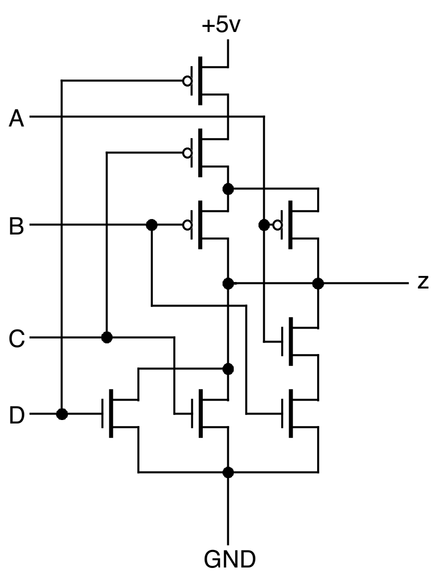


CMOS Logic vs Bipolar Logic

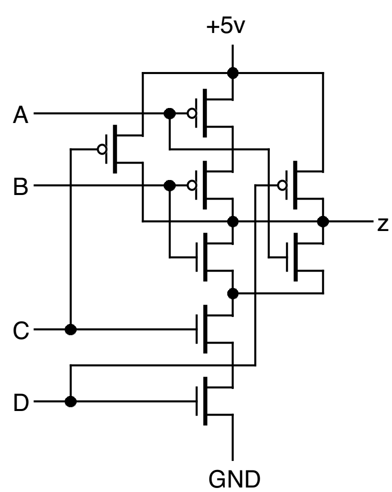
- MOSFET transistors are easier to miniaturize
- CMOS logic has lower current drain
- CMOS logic is easier to manufacture

Due: October 30, 2003

1. (20 points) Draw schematics for the following functions using AND, OR and NOT gates. (Do not simplify the formulas.)
   (a) $X(Y+Z)$
   (b) $\overline{X} + \overline{Y}\,\overline{Z}$
   (c) $\overline{X(Y+Z)}$
   (d) $W(X+YZ)$
2. (10 points) Question A.3, page 493, Murdocca & Heuring
3. (10 points) Prove the Consensus Theorem $AB + \overline{A}C + BC = AB + \overline{A}C$ using the postulates and theorems of Boolean algebra (except the Consensus Theorem itself) in Table A-1 (p. 451). *Hint:* use absorption creatively.
4. (40 points) For each CMOS circuit below,
   (a) Provide a truth table for the circuit's function.
   (b) For diagram (a), write down the Sum-of-Products (SOP) Boolean formula for the truth table. For diagram (b), wrtie down the Product-of-Sums (POS) Boolean formula.
   (c) Simplify the SOP or POS formula using the postulates and theorems of Boolean Algebra (p. 451). *Show all work.*
   (d) Draw the logic diagram of the simplified formula using AND, OR, NAND, NOR and NOT gates.



(a)



(b)

Next time

- Circuits for Addition