Of Pizza, Pirates and Preferences.

You've been captured by pirates and, while stumbling along the beach digging for buried treasure, the pirates chance upon a whole pizza with all the toppings. The pirates are really sick of their diet of salt pork and stale bread and are eager to divvy up the pizza for lunch. However, pirates are petty, greedy, suspicious, insanely jealous, prone to violence and generally badly behaved (that's how they wound up as pirates). Thus, they immediately begin squabbling over how to share the pizza. If swords are drawn, you might get hurt. How can you help divide the pizza fairly?

One complication in this problem is that the pirates do not have the same preferences for pizza toppings. One pirate might like pepperoni while another one anchovies. Being mutually suspicious, they don't want to reveal their own preferences and would not believe each other anyway.

In the case of two pirates, a simple cut-and-choose procedure provides the solution. One pirate cuts the pizza into two pieces that he thinks are equal according to his own preferences. The other pirate then chooses one of the pieces. The first pirate gets a piece that he thinks is half the pizza. The second pirate gets to choose first, so he can pick the piece that he thinks is bigger. So, both pirates are happy. Note that the procedure does not require either pirate to reveal their preferences.

Too bad you've been captured by 3 pirates: Alan, Bob and Charles. (These are the pirates' real names. They have much more fearsome nicknames among pirates.) Reaching into your dim memories of a Discrete Math class, you suggest that the pirates follow this procedure:

FairShare Procedure

1. Alan divides the pizza into three pieces that he thinks are equal (according to his own preferences).

2. Bob takes the largest piece (by his own preferences) of the three and trims the piece to make it equal to the second largest piece. The part that was removed from the largest piece, which we'll call "the extra piece," will be set aside for now.

3. Charles picks one of the three pieces that he thinks is largest.

4. Bob picks next, except that if the trimmed piece was not chosen by Charles, then Bob must take it.

5. Alan takes the last piece.

6. Bob and Charles share the extra piece using cut-and-choose. Charles cuts the extra piece into two equal pieces. Bob picks.

Claim: After the FairShare procedure, each pirate receives a fair share of the pizza — i.e., each pirate receives at least one-third of the pizza according to his own preferences.

Proof: Alan receives a fair share because he always gets one of the pieces that he cut and never the trimmed piece. Since he thinks the three original pieces are equal, he received one-third of the whole pizza. (Here, and subsequently, each pirate considers the size of a piece of pizza according to his own preferences.)

Bob made the two largest pieces equally large and he gets one of these two, since he gets to pick second. Thus, not counting the extra piece, he gets at least one-third of the pizza. Plus, he gets at least half of the extra piece, so he has at least a third of the whole pizza.

Charles picked first and picked the largest piece according to his own preferences. Thus, like Bob, he has at least one-third of the pizza, not counting the extra piece. Since he cuts the extra piece, he gets at least half of that. Thus, Charles also received at least a third of the whole pizza. □

So all the pirates should be happy, right? No. Alan is unhappy, because even though he gets a third of the pizza, he might think that Bob or Charles received a bigger share. For example, if Charles picked the untrimmed piece, according to Alan's preferences, he already has a third. However, Charles gets a portion of the extra piece in Step 6, so Alan thinks that Charles' share is bigger than his. Squabbling would ensue. You need to modify Step 6.

BiggestShare Procedure

Note: Steps 1 through 5 are the same as in FairShare. *Only Step 6 has changed.*

1. Alan divides the pizza into three pieces that he thinks are equal (according to his own preferences).
2. Bob takes the largest piece (by his own preferences) of the three and trims the piece to make it equal to the second largest piece. The part that was removed from the largest piece, which we'll call "the extra piece," will be set aside for now.
3. Charles picks one of the three pieces that he thinks is largest.
4. Bob picks next, except that if the trimmed piece was not chosen by Charles, then Bob must take it.
5. Alan takes the last piece.
6. Dividing the extra piece.
   (a) Let X be either Bob or Charles whichever one received the untrimmed piece.
   Let Y be either Bob or Charles whichever one received the trimmed pieces.
   (b) X cuts the extra piece into three equal portions according to his own preferences.
   (c) Y picks first, then Alan, then X.

Claim: After the BiggestShare procedure, each pirate receives the biggest share of the pizza — i.e., each pirate thinks that the other two pirates received an equal or smaller share of the whole pizza than he did.

Proof: We'll say a pirate is *jealous* if he thinks another pirate has a bigger share. Let's show that none of the pirates are jealous.

As before, Alan got a piece from his original cut. So, according to his preferences, that's already one third of the pizza. From his perspective, X also got a third and Y was short-changed. In fact, Alan wouldn't mind if Y got all of the extra piece. So, Alan is not jealous of the fact that Y got to pick a portion of the extra piece before he did. Since Alan picked from the extra piece before X, Alan is not jealous of either X or Y.

Charles got to pick first in Step 3. So, from his perspective, he has the largest piece, not counting the extra piece. If Charles picked the untrimmed piece, he is X and cut the extra piece into three equal portions. So, in the end, he still has the largest share of the whole pizza. If Charles picked the trimmed piece, he is Y and he picked the a portion of the extra piece first. Since he got to pick first both times, he must think he has the largest portion of the whole pizza. Thus, Charles is not jealous of Alan or Bob.

At the end of Step 4, Bob got one of the two pieces that he thinks is tied for largest. So, he thinks that he has as much as Charles. He might have more or the same as Alan. If Bob got an untrimmed piece, then he is X and he cut the extra piece into three equal portions. So, he still thinks he has as much as Charles and at least as much as Alan. If Bob got the trimmed piece, then he is Y and got to pick a portion of the extra piece first. In either case, Bob is not jealous of Alan or Charles. $\square$

You carefully explain all this to the three pirates and make sure that they understand the proof. They agree to follow the BiggestShare procedure. The pirates are so happy that they decide to release you from captivity.

Questions

1. Argue that any procedure that guarantees that each pirate receives the biggest share also guarantees that each pirate receives a fair share.
2. A mathematically rigorous argument should make its assumptions explicit which we have not done here. List 2 assumptions about pizzas, pirates or preferences that are needed to make the BiggestShare procedure work. In each case, give an example that shows that the BiggestShare procedure does not provide each pirate with the biggest share when the assumption is violated.
3. The BiggestShare procedure does not necessarily provide the best possible division of the pizza. Describe a scenario (including the pizza and the pirates' preferences) where you can give a share of the pizza to each pirate such that each pirate receives a share that is larger (according to his own preferences) than he would have received using the BiggestShare procedure. Your scenario should not violate any of the assumptions needed for the BiggestShare procedure to work.