1 Set Equality (15 points)

Prove that for all sets A, B and C that the equality shown below is true. You must prove this equality by showing that every element of the set on the left hand side of the equality is also an element of the set on the right hand side, and vice versa. (I.e., do not prove this using algebraic identities.)

$$(A \cup B) - (C - A) = A \cup (B - C)$$

(LHS $\subseteq$ RHS)

If $x \in (A \cup B) - (C - A)$, then $x \in A \cup B$ and $x \notin C - A$

Thus, $x \notin C$ or $x \in A$

Case 1: $x \in A$. Then $x \in A \cup (B - C)$

Case 2: $x \notin A$. Then $x \in B$, since $x \in A \cup B$.

Also, $x \notin A \Rightarrow x \notin C$, since $x \notin C - A$.

[by De Morgan's: $x \notin C-A \Leftrightarrow x \notin C \cap \bar{A} \Leftrightarrow x \in \overline{C \cap \bar{A}} \Leftrightarrow x \in \bar{C} \cup A$]

Thus $x \in B - C$ and $x \in A \cup (B - C)$

In both cases, we have $x \in A \cup (B - C)$,

so $(A \cup B) - (C - A) \subseteq A \cup (B - C)$

(RHS $\subseteq$ LHS)

Suppose $x \in A \cup (B - C)$, Then, $x \in A$ or $x \in B - C$

Case 1: $x \in A$

Then $x \in A \cup B$

Also, $x \notin C - A$

Thus, $x \in (A \cup B) - (C - A)$

Case 2: $x \in B - C$

Then $x \in B$ and $x \notin C$

$x \in B \Rightarrow x \in A \cup B$

Also, $x \notin C \Rightarrow x \notin C - A$

Thus, $x \in (A \cup B) - (C - A)$.

In both cases, we have $x \in (A \cup B) - (C - A)$

$\therefore A \cup (B - C) \subseteq (A \cup B) - (C - A)$

2 Mathematical Induction (15 points)

Prove that the following induction hypothesis is true for $n \geq 1$ using mathematical induction. In the proof of the induction step, make sure that you clearly indicate:

- what you are allowed to assume from the induction hypothesis,
- which step(s) of the proof uses the induction hypothesis, and
- what you need to prove to complete the proof by induction.

Induction Hypothesis:

$$P(n): \sum_{i=1}^{n}(5i-4) \stackrel{\text{def}}{=} 1+6+11+\cdots+(5n-4) = \frac{n(5n-3)}{2}.$$

Basis Step:

$n=1$ LHS $=1$

RHS $= \frac{1(5\cdot 1-3)}{2} = \frac{2}{2} = 1$

Since LHS=RHS, $P(1)$ holds.

Induction Step:

We will show that $P(n) \Rightarrow P(n+1)$ for $n \geq 1$

We are allowed to assume $P(n)$, so we know

$$1+6+11+\cdots+(5n-4) = \frac{n(5n-3)}{2}$$

We need to show $P(n+1)$:

$$1+6+11+\cdots+(5n-4)+(5(n+1)-4) = \frac{(n+1)(5(n+1)-3)}{2}$$

We start with the LHS of $P(n+1)$

$$\underbrace{1+6+11+\cdots+(5n-4)}+(5(n+1)+4)$$

$$= \frac{n(5n-4)}{2} + (5n+5-4)$$ using the induction hypothesis

$$= \frac{1}{2}(5n^2-3n+10n+2)$$

$$= \frac{1}{2}(n+1)(5n+2)$$

$$= \frac{(n+1)(5n+5-3)}{2}$$

$$= \frac{(n+1)(5(n+1)-3)}{2}$$ this is the RHS of $P(n+1)$.

3 Counting (15 points)

For each of these questions, *you must show your work and explain your answer. Answers that consist of a single number will not receive very much credit.* Answers involving factorials, combinations and permutations do not need to be expanded (that is, you may have terms such as 5!, $C(14, 2)$ or $P(14, 3)$ in your final answer).

a. (5 points) A standard deck of 52 playing cards has 4 suits (spade, heart, diamond and club) and, in each suit, 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A). Suppose you are dealt 10 cards, which of the following statements must be true? Explain your answer.

   (1) You have at least 2 cards in each suit.

   (2) You have 3 cards in one of the 4 suits.

   (3) You do not have more than 4 cards in any suit.

Statement (2) must be true.

Suppose not. Then you have ≤ 2 cards in each suit. That means you have ≤ 8 cards total (since there are 4 suits). This contradicts having 10 cards dealt to you.

b. (5 points) In the game of bridge, each player is dealt 13 cards. Bridge players talk about the *distribution* of their cards — this is the number of cards they have in each suit (e.g., 4 spades, 2 hearts, 1 diamond and 6 clubs). How many different distributions are there in bridge?

This is a "stars & bars" (a.k.a. Girl Scout cookies) problem. We have 13 cards/stars and ~~4~~ 3 suits/bars.

\# of distributions = $C(16,3) = 560$

c. (5 points) You are holding 11 cards in your hand: 2 spades, 3 hearts, 1 diamond and 5 clubs. You want to arrange the cards so that all the cards of the same suit are adjacent to each other. How many different ways can you form such an arrangement? (Note: the individual cards are distinguishable.)

There are 4! permutations of just the suits. Within each suit, there are 2! permutations of spades, 3! of hearts, 1! of diamonds and 5! of clubs.

The total # of arrangements = $4!\,2!\,3!\,1!\,5!$

$= 34{,}560$

4 Probability & Expected Value (40 points)

For each of these questions, *you must show your work and explain your answer. Answers that consist of a single number will not receive very much credit.* Answers involving permutations and combinations can be left in terms of the permutations and combinations. Also, you do not need to multiply out the fractions and exponents in your answer. For example, if your answer is $C(4,2)\cdot(1/3)^2\cdot(2/3)^3$ you do not need to convert this to 48/243 or to 0.1975....

a. (5 points) Four friends play "odd man out". They each flip a fair coin. If 1 person has heads and the other 3 have tails, then the person with heads is the odd man. Similarly, if 1 person has tails and the other 3 have heads, then the person with tails is the odd man. What is the probability of having an odd man after each person flips just once? Briefly explain your answer.

This is a Bernoulli trials problem.

Probability of odd man $= 4\left[\left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3 + \left(\frac{1}{2}\right)\left(\frac{1}{2}\right)^3\right] = \frac{1}{2}$

pick person to be the odd man

prob of heads

prob other 3 get tails

prob of tails

prob other 3 get heads

b. (5 points) Suppose the four friends in the question above continue playing "odd man out" until someone becomes the odd man. What is the expected number of times that they have to play this game? Briefly explain your answer.

For Bernoulli trials, the expected number of trials until success is $1/p$ where p is the probability of success.

Here $p = 1/2$, so expected # of games = 2.

c. (5 points) You have 36 hard-boiled eggs and 24 raw eggs that you put all in one basket. Of the hard boiled eggs, 10 are brown eggs and 26 are white. On the other hand, 8 of the raw eggs are brown and 16 are white. Suppose that you pick 2 eggs, one after the other without replacement so that each egg in the basket is picked with equal probability. What is the probability that both eggs are brown? Briefly explain your answer.

60 eggs total

18 are brown

42 are white

$$\text{Prob}[\text{2 brown eggs picked}] = \underbrace{\frac{18}{60}}_{\text{prob. that 1st egg is brown}} \cdot \underbrace{\frac{17}{59}}_{\text{prob. that 2nd egg is brown.}} \approx 0.86440677\ldots$$

d. (5 points) Continuing with the question above, you replace the eggs you picked from the basket. So, you are back to 36 hard-boiled and 24 raw eggs. Suppose you pick one egg at random and see that it is brown. What is the conditional probability that the egg you picked is also raw? Briefly explain your answer.

$$\text{Prob}(E) = \text{Prob}[\text{egg is brown}] = 18/60$$

$$\text{Prob}(F) = \text{Prob}[\text{egg is raw}] = 24/60$$

$$\text{Prob}(E \cap F) = \text{Prob}[\text{eggs is raw \& brown}] = 8/60$$

$$\text{Prob}(E|F) = \text{Prob}[\text{raw \& brown} \mid \text{brown}]$$

$$= \frac{\text{Prob}[\text{raw \& brown}]}{\text{Prob}[\text{brown}]} = \frac{8/60}{18/60} = \frac{8}{18}$$

e. (5 points) Continuing with the question above, you replace the egg you picked from the basket. So, you are back to 36 hard-boiled and 24 raw eggs. This time you pick 5 eggs at random from the basket without replacement. What is the expected number of brown eggs among the 5 eggs that you picked? Briefly explain your answer.

Prob of brown = $18/60 = 0.3$

Then, the temptation is to take $0.3 \times 5 = 1.5$,

while this actually provides the correct answer,

the explanation is dubious. Instead, consider

P_i = probability that the ith egg is brown.

Let X = # of brown eggs picked.

$$E[X] = 1 \cdot P_1 + 2 \cdot P_2 + 3 \cdot P_3 + 4 \cdot P_4 + 5 \cdot P_5$$

$$= \left(1 \cdot 241768800 + 2 \cdot 210772800 + 3 \cdot 84309120 + 4 \cdot 15422400 + 5 \cdot 1028160\right) * \frac{1}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

$$= \frac{983072160}{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14} = 1.5$$

see next page for calculations.

f. (5 points) You roll a fair 4-sided die (with sides labeled 1, 2, 3 and 4) and a fair 12-sided die (with sides labeled 1, 2, 3, ..., 12). Let E be the event that the sum of the two dice is odd and let F be the event that the sum of the two dice is divisible by 3. Are E and F independent events? Justify your answer by explaining your calculations.

four sided die

12-sided die

	1	2	3	4
1	2	3	4	5
2	3	4	5	6
3	4	5	6	7
4	5	6	7	8
5	6	7	8	9
6	7	8	9	10
7	8	9	10	11
8	9	10	11	12
9	10	11	12	13
10	11	12	13	14
11	12	13	14	15
12	13	14	15	16

$\text{Prob}[E] = 1/2$

$\text{Prob}[F] = \frac{16}{48} = \frac{1}{3}$

$\text{Prob}[E \cap F] = \frac{8}{48} = \frac{1}{6}$

$\frac{1}{2} \cdot \frac{1}{3} = \frac{1}{6}$ =

Therefore events E and F are independent.

Let P_i = Probability of getting exactly i brown eggs.

$$P_1 = C(5,1) \cdot \underbrace{\left(\frac{18}{60}\right)}_{\text{prob 1st egg is brown}} \underbrace{\left(\frac{42}{59}\right)\left(\frac{41}{58}\right)\left(\frac{40}{57}\right)\left(\frac{39}{56}\right)}_{\text{probability remaining eggs are white}} = \frac{241768800}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

also consider case where second egg is brown & others white. This prob is

$$\frac{42}{60} \cdot \frac{18}{59} \cdot \frac{41}{58} \cdot \frac{40}{57} \cdot \frac{39}{56}$$

but numerically, this is equal to 1st case. So, just multiply by $C(5,1)$

$$P_2 = C(5,2) \cdot \frac{18 \cdot 17 \cdot 42 \cdot 41 \cdot 40}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} = \frac{210772800}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

$$P_3 = C(5,3) \frac{18 \cdot 17 \cdot 16 \cdot 42 \cdot 41}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} = \frac{84309120}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

$$P_4 = C(5,4) \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 42}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} = \frac{15422400}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

$$P_5 = C(5,5) \frac{18 \cdot 17 \cdot 16 \cdot 15 \cdot 14}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56} = \frac{1028160}{60 \cdot 59 \cdot 58 \cdot 57 \cdot 56}$$

g. (5 points) A standard deck of 52 playing cards has 4 suits (spade, heart, diamond and club) and, in each suit, 13 cards (2, 3, 4, 5, 6, 7, 8, 9, 10, J, Q, K, A). You shuffle the cards so that every permutation of the 52 cards is equally likely. You make this bet with a friend. If you draw a card at random and it is a J, Q, K or A, then your friend pays you $2. If you draw a number card (2 thru 10), then you pay your friend $1. You make 5 such bets, replacing the card and re-shuffling the deck after each bet. What is the probability that you win exactly 3 of these 5 bets? Briefly explain your answer.

Bernoulli trials

$$\underbrace{C(5,3)}_{\text{Choose 3 bets to win}} \cdot \underbrace{\left(\frac{4}{13}\right)^3}_{\text{prob that you win}} \cdot \underbrace{\left(\frac{9}{13}\right)^2}_{\text{prob that you lose}} \approx 0.1396\ldots$$

h. (5 points) Continuing with the question above, what is your expected winnings (or losses) from the 5 bets? Briefly explain your answer.

Use linearity of expectations.

Let X_i = amount won or lost in ith bet.

Let $X = X_1 + X_2 + X_3 + X_4 + X_5$

$$E[X_i] = \underbrace{2}_{\text{win \$2}} \cdot \underbrace{\left(\frac{4}{13}\right)}_{\text{prob of win}} + \underbrace{(-1)}_{\text{lose \$1}} \cdot \underbrace{\frac{9}{13}}_{\text{prob of losing.}} = \frac{-1}{13}$$

does not depend on i.

$$E[X] = E[X_1] + E[X_2] + E[X_3] + E[X_4] + E[X_5]$$
$$= 5 \cdot (-1/13) = -5/13$$

5 Equivalence Relations (10 points)

For each of the following relations, state whether the relation is reflexive, symmetric, antisymmetric, transitive and also whether it is an equivalence relation. The sets $\mathbb{N}$ and $\mathbb{R}$ are respectively the natural numbers and the real numbers. *Briefly justify your answer.*

a. (5 points) A relation R_1 on $\mathbb{R}$: $R_1 = \{ (x, y) \mid x^2 + y^2 = 1 \}$.

reflexive? No, $(1,1) \notin R_1$ since $1^2 + 1^2 \neq 1$

symmetric? Yes, since addition is commutative

antisymmetric? No, $(0,1)$ and $(1,0)$ are in R_1, since $0^2+1^2=1$ and $1^2+0^2=1$, but $0 \neq 1$

transitive? No. $(1,0) \in R_1$ and $(0,1) \in R_1$ but $(1,1) \notin R_1$

equivalence relation? No, not reflexive & not transitive

b. (5 points) A relation R_2 on $\{ X \mid X \subseteq \mathbb{N} \}$: $R_2 = \{ (X, Y) \mid X \subseteq Y \}$.

reflexive? Yes, $X \subseteq X$ is true for sets.

symmetric? No, $X = \{1\}$ & $Y = \{1,2\}$. $X \subseteq Y$, but $Y \nsubseteq X$

antisymmetric? Yes, $X \subseteq Y$ & $Y \subseteq X \Rightarrow X = Y$ is the definition of set equality.

transitive?

Yes, $X \subseteq Y$ & $Y \subseteq Z \Rightarrow X \subseteq Z$.

equivalence relation?

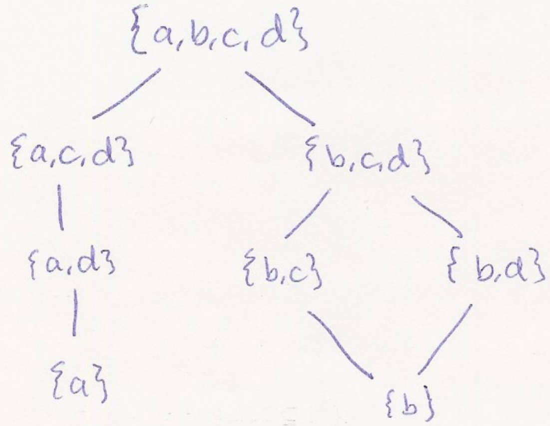
No, not symmetric.

6 Partial Orders (10 points)

Let $A = \{\{a\}, \{b\}, \{b, c\}, \{a, c, d\}, \{a, d\}, \{b, d\}, \{b, c, d\}, \{a, b, c, d\}\}$. Each element of A is a subset of $\{a, b, c, d\}$. Let R be the subset relation on A, that is:

$$R = \{\ (X, Y) \mid X \in A,\ Y \in A \text{ and } X \subseteq Y\ \}.$$

a. (4 points) Draw a Hasse diagram for R.



b. (2 points) List two incomparable elements in R.

{b,c} and {b,d} are incomparable

c. (2 points) What are the minimal elements of this partial order?

{a} and {b}

d. (2 points) Does R have a ~~greatest~~ maximum element? Why or why not?

Yes, {a,b,c,d} is the maximum since every element of A is contained in {a,b,c,d}.