Section 3: Linear, Homogeneous Recurrence Relations

- So far, we have seen that certain simple recurrence relations can be solved merely by interative evaluation and keen observation.
- In this section, we seek a more methodical solution to recurrence relations.
- In particular, we shall introduce a general technique to solve a broad class of recurrence relations, which will encompass those of the last section as well as the tougher Fibonnaci relation.

Linear, Homogeneous Recurrence Relations with Constant Coefficients

- If A and B ($\neq 0$) are constants, then a recurrence relation of the form: $a_k = Aa_{k-1} + Ba_{k-2}$
  is called a *linear*, *homogeneous*, *second order*, *recurrence relation with constant coefficients*.
- We will use the acronym LHSORRCC.
- <u>Linear:</u> All exponents of the a_k's are 1;
- <u>Homogeneous:</u> All the terms have the same exponent.
- <u>Second order:</u> a_k depends on a_{k-1} and a_{k-2};

Higher Order Linear, Homogeneous Recurrence Relations

- If we let C_1, C_2, C_3, ..., C_n be constants ($C_{last} \neq 0$), we can create LHRRCC's of arbitrary order.
- As we shall see, the techniques the book develops for second order relations generalizes nicely to higher order recurrence relations.
- <u>Third order:</u> $a_k = C_1 a_{k-1} + C_2 a_{k-2} + C_3 a_{k-3}$
- <u>Fourth order:</u> $a_k = C_1 a_{k-1} + C_2 a_{k-2} + C_3 a_{k-3} + C_4 a_{k-4}$
- <u>*n*th order:</u> $a_k = C_1 a_{k-1} + C_2 a_{k-2} + \ldots + C_n a_{k-n}$;

Solving LHSORRCC's

- Let's start with the second order case before we generalize to higher orders.
- Definition: Given $a_k = Aa_{k-1} + Ba_{k-2}$, the *characteristic equation* of the recurrence relation is $x^2 = Ax + B$, and the *characteristic polynomial* of the relation is $x^2 - Ax - B$.
- Theorem: Given $a_k = Aa_{k-1} + Ba_{k-2}$, if s,t,C,D are non-zero real numbers, with $s \neq t$, and s,t satisfy the characteristic equation of the relation, then its General Solution is $a_n = C(s^n) + D(t^n)$.

An Example

- Let $a_k = 5a_{k-1} - 6a_{k-2}$. Find the general solution.
- The relation has characteristic equation:

$$x^2 = 5x - 6,$$

so $$x^2 - 5x + 6 = 0$$

hence $$(x - 2)(x - 3) = 0$$

implying either $(x - 2) = 0$ or $(x - 3) = 0$

thus $$x = 2,3$$

- General Solution is $a_n = C(2^n) + D(3^n)$.

Finding Particular Solutions

- Once we have found the general solution to a recurrence relation, if we have a sufficient number of initial conditions, we can find the *particular solution*.
- This means we find the values for the arbitrary constants C and D, so that the solution for the recurrence relation takes on those initial conditions.
- The required number of initial conditions is the same as the order of the relation.

An Example

- For the last example, we found the recurrence relation $a_k = 5a_{k-1} - 6a_{k-2}$ has general solution $a_n = C(2^n) + D(3^n)$. Find the particular solution when $a_0 = 9$ and $a_1 = 20$.

$$a_0 = C(2^0) + D(3^0) = C + D = 9$$

$$a_1 = C(2^1) + D(3^1) = 2C + 3D = 20, \text{ so}$$

$$2C + 2D = 18$$

$$2C + 3D = 20, \text{ so } D = 2 \text{ and } C = 7.$$

Therefore, the particular solution is:

$$a_n = 7(2^n) + 2(3^n).$$

Generalizing These Methods

- We can extend these techniques to higher order LHRRCC quite naturally. Suppose we have a LHRRCC whose charateristic poylnomial has roots x = 2, –3,5,7,11, and 13. Then its general solution is:

$$a_n = C2^n + D(-3)^n + E5^n + F7^n + G11^n + H13^n.$$

- Moreover, if we have initial conditions specified for a_0, a_1, a_2, a_3, a_4, and a_5, we can plug them into the general solution and get a 6×6 system of equations to solve for C, D, E, F, G, and H.

Solving The Fibonacci Relation

- Solve: $a_n = a_{n-1} + a_{n-2}$ when $a_0 = 1$ and $a_1 = 1$.
- Solution: In this case, the characteristic polynomial is $x^2 - x - 1$, which doesn't factor nicely. We turn to the quadratic formula to find the roots.
- Quadratic Formula: If $ax^2 + bx + c = 0$, then
$$x = [-b \pm \sqrt{(b^2 - 4ac)}]/2a.$$
- In our case, we have $a = 1$, $b = -1$ and $c = -1$, so $x = [-(-1) \pm \sqrt{((-1)^2 - 4(1)(-1))}]/2(1) = (1 \pm \sqrt{5})/2$.
- Thus $a_n = C[(1 + \sqrt{5})/2]^n + D[(1 - \sqrt{5})/2]^n$.

Solving The Fibonacci Relation (*cont'd.*)

- If we apply the initial conditions $a_0 = 1$ and $a_1 = 1$ to $a_n = C[(1 + \sqrt{5})/2]^n + D[(1 - \sqrt{5})/2]^n$, we get:

  $a_0 = C + D = 1$

  $a_1 = [(1 + \sqrt{5})/2]C + [(1 - \sqrt{5})/2]D = 1$, yielding

  $C = (1 + \sqrt{5})/(2\sqrt{5})$ and $D = -(1 - \sqrt{5})/(2\sqrt{5})$.

- Therefore $a_n = [(1 + \sqrt{5})/(2\sqrt{5})][(1 + \sqrt{5})/2]^n + [-(1 - \sqrt{5})/(2\sqrt{5})][(1 - \sqrt{5})/2]^n$

- This simplifies to

  $a_n = (1/\sqrt{5})\{[(1 + \sqrt{5})/2]^{n+1} - [(1 - \sqrt{5})/2]^{n+1}$

Single Root Case

- So far, our technique for solving LHSORRCCs has depended on the fact that the two roots of the characteristic polynomial are distinct.
- This is not always the case, however. We can find that a polynomial has only one root, s, whenever the polynomial factors as $(x - s)^2$.
- In this case, our solution takes on a special variant to ensure "linear independence" of the solutions.
- <u>Theorem:</u> If an LHSORRCC has a repeated root s, then the general solution is $a_n = (A + Bn)s^n$.

Single Root Case Example

- Find the general solution of $a_n - 6a_{n-1} + 9a_{n-2} = 0$.
- This LHSORRCC has a characteristic polynomial equation of $x^2 - 6x + 9 = 0$, so $(x - 3)^2 = 0$, which yields the sole root $x = 3$.
- Therefore, the general solution is $a_n = (A + Bn)3^n$.
- If we add initial conditions $a_0 = 2$ and $a_1 = 21$, we get: $a_0 = (A + B(0))3^0 = A = 2$, and
  $a_1 = (A + B(1))3^1 = 3(A + B) = 3(2 + B) = 21$,
  so $2 + B = 7$, hence $B = 5$.
- Therefore the particular solution is $a_n = (2 + 5n)3^n$

Higher Order Repeated Root Case

- This method of building up the "coefficient" when the variable part degenerates because of repeated roots extends nicely to higher order problems as well.
- Example: If a LHRRCC has characteristic polynomial with roots x = 7,7,7,7,7,7,7,9,9,9 then its general solution is:
- $a_n = (A + Bn + Cn^2 + Dn^3 + En^4 + Fn^5 + Gn^6)7^n$
  $+ (H + In + Jn^2)9^n$.
- How many IC are needed for a particular solution?

Summary

- Our general technique for solving LHRRCCs is a two-step process.
- Step 1: Find the roots of the characteristic polynomial and use them to develop the general solution.

  How do I find roots of polynomials?
- Step 2: Use the initial conditions to make and solve a system of linear equations that determine the arbitrary constants in the general solution to get the particular solution.

  How do I solve systems of linear equations?

Board Example #1

- Given the recurrence relation $a_n = 4a_{n-1} - 3a_{n-2}$, find a_{999} when $a_0 = 5$ and $a_1 = 7$.

Board Example #2

- Given the recurrence relation $a_n = 4a_{n-1} - 4a_{n-2}$, find a_{999} when $a_0 = 5$ and $a_1 = 7$.

Board Example #3

- What is the general solution for the LHRRCC whose characteristic polynomial is:

$$(x + 5)^6(x - 3)^4(x + 8)^2$$

Board Example #4

- Given the LHRRCC $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$, find a_{999} when $a_0 = 17$, $a_1 = 14$, and $a_2 = 110$.

Validity of the General Solution I

Prove: If $Aa_n + Ba_{n-1} + Ca_{n-2} = 0$, and $s \neq t$ satisfy $Ax^2 + Bx + C = 0$, then $a_k = Ms^k + Nt^k$ satisfies the relation.

Proof: Let $Aa_n + Ba_{n-1} + Ca_{n-2} = 0$, and $s \neq t$ satisfy $Ax^2 + Bx + C = 0$. Thus:

$$As^2 + Bs + C = At^2 + Bt + C = 0.$$

Now, $a_n = Ms^n + Nt^n$, $a_{n-1} = Ms^{n-1} + Nt^{n-1}$, and $a_{n-2} = Ms^{n-2} + Nt^{n-2}$ hence $Aa_n + Ba_{n-1} + Ca_{n-2}$

$$= A(Ms^n + Nt^n) + B(Ms^{n-1} + Nt^{n-1}) + C(Ms^{n-2} + Nt^{n-2})$$

$$= M(As^n + Bs^{n-1} + Cs^{n-2}) + N(At^n + Bt^{n-1} + Ct^{n-2})$$

$$= Ms^{n-2}(As^2 + Bs + C) + Nt^{n-2}(At^2 + Bt^{n-1} + C) = 0.$$ QED

Validity of the General Solution II

<u>Prove:</u> If $Aa_n + Ba_{n-1} + Ca_{n-2} = 0$, and s is the only solution of $Ax^2 + Bx + C = 0$, then $a_k = (P + Qk)s^k$ satisfies the relation.

<u>Proof:</u> Let $Aa_n + Ba_{n-1} + Ca_{n-2} = 0$, and s be the only solution of $Ax^2 + Bx + C = 0$, so $As^2 + Bs + C = 0$.

Now, $a_n = (P + Qn)s^n$, $a_{n-1} = [P + Q(n-1)]s^{n-1}$, and $a_{n-2} = [P + Q(n-2)]s^{n-2}$ hence $Aa_n + Ba_{n-1} + Ca_{n-2}$

$$= A(P + Qn)s^n + B[P + Q(n-1)]s^{n-1} + C[P + Q(n-2)]s^{n-2}$$

$$= P(As^n + Bs^{n-1} + Cs^{n-2}) + Q[Ans^n + B(n-1)s^{n-1} + C(n-2)s^{n-2}]$$

Validity of the General Solution II

Thus, $Aa_n + Ba_{n-1} + Ca_{n-2}$

$= Ps^{n-2}(As^2 + Bs + C) + Q(Ans^n + Bns^{n-1} - Bs^{n-1}$
$+ Cns^{n-2} - 2Cs^{n-2})$

$= Qns^{n-2}(As^2 + Bs + C\ ) + Qs^{n-2}(-Bs - 2C)$
$= Qs^{n-2}(-Bs - 2C) = 0$?????

However, since s is the only root of the characteristic polynomial, from the Quadratic Formula, we have that $(B^2 - 4AC) = 0$ and $s = -B/2A$.

Thus $(-Bs - 2C) = -B(-B/2A) - 2C$
$= B^2/2A - 2C(2A/2A) = (B^2 - 4AC)/2A = 0$. QED